A TWO-SAMPLE TEST FOR A LINEAR HYPOTHESIS WHOSE POWER IS INDEPENDENT OF THE VARIANCE

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1. Introduction. In a paper in the *Annals of Mathematical Statistics*, Dantzig [1] proves that, for a sample of fixed size, there does not exist a test for Student's hypothesis whose power is independent of the variance. Here, a two-sample test with this property will be presented, the size of the second sample depending upon the result of the first. The problem of determining confidence intervals, of preassigned length and confidence coefficient, for the mean of a normal distribution with unknown variance is solved by the same procedure. These considerations including the non-existence of a single-sample test whose power is independent of the variance, are extended to the case of a linear hypothesis. In order to make the power of a test or the length of a confidence interval exactly independent of the variance, it appears necessary to waste a small part of the information. Thus, in practical applications, one will not use a test with this property, but rather a test which is uniformly more powerful, or an interval of the same length, whose confidence coefficient is a function of σ, but always greater than the desired value, the difference usually being slight, at the same time reducing the expected number of observations by a small amount.

Any two sample procedure, such as that discussed in this paper, can be considered a special case of sequential analysis developed by Wald [5].

The problem of whether these tests and confidence intervals are in any sense optimum is unsolved. It is difficult even to formulate a definition of an optimum among sequential tests of a hypothesis against multiple alternatives. However it is shown that, if the variance and initial sample size are sufficiently large, the expected number of observations differs only slightly from the number of observations required for a single-sample test when the variance is known. It also seems likely that the confidence intervals do possess some optimum property among the class of all two-sample procedures.

Although Student's hypothesis is a special case of a linear hypothesis, it is treated separately, because it illustrates the basic idea without any complicated notation or new distributions. The test for Student's hypothesis involves the use only of Student's distribution, even for the power of the test, while the power function of the test proposed here for a linear hypothesis involves a new type of non-central F-distribution.

The notation χ^2_n is used as a generic symbol for a random variable equal to the sum of squares of n independently normally distributed random variables with mean 0 and variance 1, i.e., χ^2_n has the χ^2 distribution with n degrees of freedom,

$$P\{\chi_n^2 < T\} = \frac{1}{(\sqrt{2})^n \Gamma(\frac{1}{2}n)} \int_0^T e^{-\frac{1}{2}u} u^{\frac{1}{2}n-1}\, du \qquad \text{for } T \geq 0$$
$$= 0 \qquad \text{for } T \leq 0.$$

The notation t_n is used as a generic symbol for $\frac{x\sqrt{n}}{\chi_n}$, where x is normally distributed with mean 0 and variance 1, independently of χ_n^2, i.e., t_n has the distribution of Student's t with n degrees of freedom,

$$P\{t_n < t\} = \frac{\Gamma(\frac{1}{2}(n+1))}{\sqrt{n\pi}\,\Gamma(\frac{1}{2}n)} \int_{-\infty}^{t} \left(1 + \frac{z^2}{n}\right)^{-\frac{1}{2}(n+1)} dz.$$

$F_{m,n}$ is a generic symbol for a random variable of the form $F_{m,n} = n\chi_m^2/m\chi_n^2$, the numerator and denominator being independently distributed, i.e., $F_{m,n}$ has the distribution of an F-ratio with m and n degrees of freedom,

$$P\{F_{m,n} < T\} = \frac{\Gamma(\frac{1}{2}(m+n))}{\Gamma(\frac{1}{2}m)\Gamma(\frac{1}{2}n)} \int_0^T \left(\frac{m}{n}\right)^{\frac{1}{2}m} F^{\frac{1}{2}m-1} \left(1 + \frac{m}{n}F\right)^{-\frac{1}{2}(m+n)} dF.$$

A symbol of the above type with an additional subscript α denotes the upper $100\alpha\%$ significance level, e.g., $t_{n,\alpha}$ is defined by

$$P\{t_n > t_{n,\alpha}\} = \alpha.$$

The symbol $E\{x \mid Q(x)\}$ denotes the set of all x such that the condition $Q(x)$ holds. This should not be confused with $E(x \mid T)$, which denotes the expected value of a random variable x, given the conditions T.

The size of a critical region is the probability that the sample point will lie within the region under the null hypothesis. The terms length and volume, as applied to confidence regions are used in the ordinary geometrical sense.

2. The test for Student's hypothesis. Suppose x_i, $i = 1, 2, \cdots$ are independently normally distributed with mean ξ and variance σ^2. We wish to test the hypothesis $\xi = \xi_0$, the power of the test to depend only upon $\xi - \xi_0$, not upon σ^2. For this purpose we define a statistic t' as follows. A sample of n_0 observations, $x_1 \cdots x_{n_0}$ is taken, and the sample estimate, s^2, of the variance computed by

$$s^2 = \frac{1}{n_0 - 1}\left\{\sum_1^{n_0} x_i^2 - \frac{1}{n_0}\left(\sum_1^{n_0} x_i\right)^2\right\}. \tag{1}$$

Then n is defined by

$$n = \max\left\{\left[\frac{s^2}{z}\right] + 1, n_0 + 1\right\}, \tag{2}$$

where z is a previously specified positive constant, $[q]$ denoting the smallest integer less than q. Additional observations, $x_{n_0+1}, \cdots, x_n$ are taken, and, in accordance with an initially specified rule depending only upon s^2, real numbers a_i, $i = 1 \cdots n$ are chosen in such a way that

$$\sum_1^n a_i = 1, \qquad a_1 = a_2 = \cdots = a_{n_0} \tag{3}$$
$$s^2 \sum_1^n a_i^2 = z.$$

This is clearly possible since

$$\min \sum_1^n a_i^2 = \frac{1}{n} \leq \frac{z}{s^2} \quad \text{by (2)}, \tag{4}$$

the minimum being taken subject to the conditions

$$\sum_1^n a_i = 1, \qquad a_1 = a_2 = \cdots = a_{n_0}.$$

Then t' is defined by

$$\begin{aligned} t' &= \frac{\sum_1^n a_i x_i - \xi_0}{\sqrt{z}} = \frac{\sum_1^n a_i(x_i - \xi)}{\sqrt{z}} + \frac{\xi - \xi_0}{\sqrt{z}} \\ &= u + \frac{\xi - \xi_0}{\sqrt{z}}, \end{aligned} \tag{5}$$

where

$$u = \frac{\sum_1^n a_i(x_i - \xi)}{\sqrt{z}}. \tag{6}$$

Then u has the distribution of Student's t with $n_0 - 1$ degrees of freedom, regardless of the value of σ^2. For $(n_0 - 1)s^2/\sigma^2$ has the distribution of $\chi^2_{n_0-1}$ and the conditional distribution of $\frac{1}{\sqrt{z}} \sum_1^n a_i(x_i - \xi) = u$, given s, is normal with mean 0 and variance $\sigma^2 \Sigma a_i^2/z = \sigma^2/s^2$. But the usual form of a random variable t_{n_0-1} is $t_{n_0-1} = y/s$, y being normally distributed with mean 0 and variance σ^2, and $(n_0 - 1)s^2/\sigma^2$ having the distribution of $\chi^2_{n_0-1}$, independent of y. Thus the conditional distribution of u, given s, is normal with mean 0 and variance σ^2/s^2, so that t_{n_0-1} and u have the same distribution.

This theorem can be used to obtain an unbiased test for the hypothesis H_0 that $\xi = \xi_0$, the power being independent of σ^2, which is supposed unknown. Let α be the desired size of the critical region and let $t_{n_0-1,\alpha/2}$ be such that

$$P\{t_{n_0-1} > t_{n_0-1,\alpha/2}\} = \frac{\alpha}{2}. \tag{7}$$

Then if we reject H_0 whenever

$$\left| \frac{\sum_1^n a_i x_i - \xi_0}{\sqrt{z}} \right| > t_{n_0-1,\alpha/2}, \tag{8}$$

we obtain an unbiased test of H_0, whose power function is $1 - \beta(\xi)$ where

$$\beta(\xi) = P\left\{-t_{n_0-1,\alpha/2} + \frac{\xi_0 - \xi}{\sqrt{z}} < t_{n_0-1} < t_{n_0-1,\alpha/2} + \frac{\xi_0 - \xi}{\sqrt{z}}\right\}. \tag{9}$$

The fact that the test is unbiased follows immediately from the symmetry and unimodality of the t distribution.

If we wish to test the hypothesis $H_0:\xi = \xi_0$ against one-sided alternatives $\xi > \xi_0$, the procedure is similar. The critical region of size α is defined by

$$\frac{\sum_1^n a_i x_i - \xi_0}{\sqrt{z}} > t_{n_0-1,\alpha} \tag{10}$$

and the power function is

$$1 - \beta(\xi) = P\left\{t_{n_0-1} > t_{n_0-1,\alpha} + \frac{\xi_0 - \xi}{\sqrt{z}}\right\}. \tag{11}$$

A confidence interval for ξ, of predetermined length l and confidence coefficient $1 - \alpha$ can be obtained by selecting z so that

$$\begin{aligned} 1 - \alpha &= P\left\{-\frac{l}{2\sqrt{z}} < t_{n_0-1} < \frac{l}{2\sqrt{z}}\right\} \\ &= P\left\{-\frac{l}{2\sqrt{z}} < \frac{\sum_1^n a_i(x_i - \xi)}{\sqrt{z}} < \frac{l}{2\sqrt{z}}\right\} \\ &= P\left\{\xi - \frac{l}{2} < \sum_1^n a_i x_i < \xi + \frac{l}{2}\right\} \\ &= P\left\{\left|\sum_1^n a_i x_i - \xi\right| < \frac{l}{2}\right\} \\ &= P\left\{\sum_1^n a_i x_i - \frac{l}{2} < \xi < \sum_1^n a_i x_i + \frac{l}{2}\right\}, \end{aligned} \tag{12}$$

where ξ is the true mean of the distribution. Thus $(\Sigma a_i x_i - l/2, \Sigma a_i x_i + l/2)$ is the desired confidence interval.

In the above tests and confidence intervals, the distribution of the required number of observations, n, is

$$P\{n = n_0 + 1\} = P\left\{\frac{s^2}{z} \leq n_0 + 1\right\}$$

$$
\begin{aligned}
(13) \qquad &= P\{(n_0 - 1)s^2/\sigma^2 < (n_0 + 1)(n_0 - 1)z/\sigma^2\} = P\{\chi^2_{n_0-1} < y\} \\
&= \frac{1}{(\sqrt{2})^{n_0-1}\Gamma(\frac{1}{2}(n_0 - 1))} \int_0^y e^{-\frac{1}{2}u} u^{\frac{1}{2}(n_0-3)}\, du,
\end{aligned}
$$

where $y = (n_0^2 - 1)\, z/\sigma^2$,

$$
\begin{aligned}
P\{n = \nu\} &= P\left\{\nu < \frac{s^2}{z} + 1 \leq \nu + 1\right\} \\
(14) \qquad &= P\{(\nu - 1)(n_0 - 1)z/\sigma^2 < \chi^2_{n_0-1} < \nu(n_0 - 1)z/\sigma^2\} \\
&= \frac{1}{(\sqrt{2})^{n_0-1}\Gamma(\frac{1}{2}(n_0 - 1))} \int_{(\nu-1)(n_0-1)z/\sigma^2}^{\nu(n_0-1)z/\sigma^2} e^{-\frac{1}{2}u} u^{\frac{1}{2}(n_0-3)}\, du,
\end{aligned}
$$

for integral $\nu > n_0 + 1$, all other values being impossible. Thus the expected number of observations, $E(n)$, satisfies the inequalities

$$
\begin{aligned}
&\frac{1}{(\sqrt{2})^{n_0-1}\Gamma(\frac{1}{2}(n_0 - 1))}\left\{\int_0^y (n_0 + 1)e^{-\frac{1}{2}u} u^{\frac{1}{2}(n_0-3)} + \int_y^\infty e^{-\frac{1}{2}u} u^{\frac{1}{2}(n_0-3)} \frac{\sigma^2 u}{z(n_0 - 1)}\, du\right\} \\
&\quad < E(n) \\
(15) \qquad &\quad < \frac{1}{(\sqrt{2})^{n_0-1}\Gamma(\frac{1}{2}(n_0 - 1))}\left\{\int_0^y (n_0 + 1)e^{-\frac{1}{2}u} u^{\frac{1}{2}(n_0-3)}\, du + \int_y^\infty e^{-\frac{1}{2}u} u^{\frac{1}{2}(n_0-3)}\right. \\
&\qquad\qquad \left. \cdot\left(\frac{\sigma^2 u}{z(n_0 - 1)} + 1\right) du\right\},
\end{aligned}
$$

which can be rewritten

$$
\begin{aligned}
&(n_0 + 1)P\{\chi^2_{n_0-1} < y\} + \frac{\sigma^2}{z} P\{\chi^2_{n_0+1} > y\} \\
(16) \qquad &\quad < E(n) < (n_0 + 1)P\{\chi^2_{n_0-1} < y\} + \frac{\sigma^2}{z} P\{\chi^2_{n_0+1} > y\} + P\{\chi^2_{n_0-1} > y\}.
\end{aligned}
$$

Consequently $E(n)$ is a function of σ^2, and can be evaluated from tables of the incomplete Γ function.

As mentioned in the introduction, these tests and confidence intervals will not be used exactly in this form, since they waste information in order to make the power of the test or the length of the confidence interval strictly independent of the variance. Instead of (2) we take a total of

$$
(17) \qquad n = \max\left\{\left[\frac{s^2}{z}\right] + 1, n_0\right\}
$$

observations, and define

$$
\begin{aligned}
t'' &= \frac{\left(\frac{1}{n}\sum_1^n x_i - \xi_0\right)\sqrt{n}}{s} \\
(18) \qquad &= \frac{\frac{1}{n}\sum_1^n (x_i - \xi)}{s}\sqrt{n} + \frac{\xi - \xi_0}{s}\sqrt{n} \\
&= u' + \frac{\xi - \xi_0}{s}\sqrt{n}.
\end{aligned}
$$

By the same reasoning as that following (6), u' has the t distribution with $n_0 - 1$ degrees of freedom. By (2)

$$n \geq s^2/z \quad \text{so that, although} \quad \left|\frac{\xi - \xi_0}{s}\sqrt{n}\right| \tag{19}$$

is a random variable,

$$\left|\frac{\xi - \xi_0}{s}\sqrt{n}\right| \geq \left|\frac{\xi - \xi_0}{\sqrt{z}}\right| \tag{20}$$

Thus, if we use

$$| t'' | > t_{n_0-1,\alpha/2} \text{ or } t'' > t_{n_0-1,\alpha} \tag{21}$$

instead of (8) or (10) respectively, we shall always increase the power of the test. Also the expected number of observations will be reduced from that in (16) by $P\{\chi^2_{n_0-1} < y\}$. Similarly if z is defined as in (12), the interval

$$\left(\frac{1}{n}\sum_1^n x_i - \frac{l}{2}, \quad \frac{1}{n}\sum_1^n x_i + \frac{l}{2}\right)$$

has length l, and the probability that it covers the true mean ξ is a function of σ, but is always greater than $1 - \alpha$, and differs only slightly from $1 - \alpha$ if $\sigma^2 > n_0 z$. Thus it can be used instead of the confidence interval (12).

From (16) it follows that

$$\overline{\lim_{\sigma\to\infty}} \left\{E(n) - \frac{\sigma^2}{z}\right\} \leq 1$$

$$\underline{\lim_{\sigma\to\infty}} \left\{E(n) - \frac{\sigma^2}{z}\right\} \geq 0,$$

the approximation $E(n) \approx \sigma^2/z$ being fair provided $\sigma^2 > zn_0$. The length of the confidence interval (12) is given by

$$l = 2t_{n_0-1,\alpha/2}\sqrt{z} \approx \frac{2\sigma t_{n_0-1,\alpha/2}}{\sqrt{E(n)}}$$

When the variance σ^2 is known, the length of the single-sample confidence interval of confidence coefficient $1 - \alpha$ obtained on the basis of n observations is given by

$$1 - \alpha = \frac{1}{\sqrt{2\pi}}\int_{-l\sqrt{n}/2\sigma}^{l\sqrt{n}/2\sigma} e^{-\frac{1}{2}x^2}\,dx$$

i.e.,

$$l = 2t_{\infty,\alpha/2}\sigma/\sqrt{n}\,.$$

Since, even for moderate values of n_0, say $n_0 \geq 30$, $t_{n_0-1,\alpha/2}$ differs only slightly from $t_{\infty,\alpha/2}$, the expected number of observations for a confidence interval of given length and confidence coefficient is only slightly larger than the fixed number of observations required in the single-sample case when the variance is known provided the variance is moderately large.

3. Distribution of a non-central F-ratio. In the extention of the above considerations to the testing of a general linear hypothesis, the power function depends on the distribution of a quantity

$$F' = \sum_1^m (q_i - c_i)^2, \tag{22}$$

where $q_i = \dfrac{x_i}{\sqrt{r}}$, x_i being independently normally distributed with mean 0 and variance 1, and r having the χ^2_n distribution, independently of the x_i. The c_i are real constants.

Let

$$\zeta = \sum_1^m c_i x_i \Big/ \sqrt{\sum_1^m c_i^2} \tag{23}$$

$$\begin{aligned}\chi^2 &= \sum_1^m (x_i - c_i\zeta)^2 = \sum_1^m x_i^2 - \zeta^2 \\ &= \sum_1^m (x_i - c_i\sqrt{r})^2 - \left(\zeta - \sqrt{r}\sqrt{\sum_1^m c_i^2}\right)^2.\end{aligned} \tag{24}$$

Now, $\sum_1^m (x_i - c_i\zeta)^2$ is a quadratic form of rank $m - 1$ since the $x_i - c_i\zeta$ are subject to one linear homogeneous restriction, namely $\sum_i c_i(x_i - c_i\zeta) = 0$. Also ζ^2 is of rank 1, and $\chi^2 + \zeta^2 = \sum_1^m x_i^2$ so that, by Cochran's Theorem, χ^2 and ζ^2 are independently distributed as χ^2_{m-1} and χ^2_1 respectively. Thus there exist $y_1 \cdots y_m$, independently normally distributed with mean 0 and variance 1 such that

$$\begin{aligned}\chi^2 &= y_2^2 + \cdots + y_m^2 \\ \zeta^2 &= y_1^2.\end{aligned} \tag{25}$$

Let $u_i = \dfrac{y_i}{\sqrt{r}}$. Then the joint distribution of $u_1 \cdots u_m$ is given by

$$\begin{aligned}P\{u_1 < \tau_1, \cdots, u_m < \tau_m\} &= \frac{1}{(\sqrt{2\pi})^m}\frac{1}{(\sqrt{2})^n\Gamma(\frac{1}{2}n)} \\ &\times \int_0^\infty e^{-\frac{1}{2}r} r^{\frac{1}{2}(n-2)}\,dr \int_{-\infty}^{\tau_1\sqrt{r}} \cdots \int_{-\infty}^{\tau_m\sqrt{r}} e^{-\frac{1}{2}\sum_1^m y_i^2}\,dy_1 \cdots dy_m.\end{aligned} \tag{26}$$

The density function is given by

$$\frac{\partial^m P\{u_1 < \tau_1, \cdots, u_m < \tau_m\}}{\partial\tau_1 \cdots \partial\tau_m}$$

$$\begin{aligned}
&= \frac{1}{(\sqrt{2\pi})^m} \frac{1}{(\sqrt{2})^n \Gamma(\frac{1}{2}n)} \int_0^\infty e^{-\frac{1}{2}r} r^{\frac{1}{2}(n-2)} r^{\frac{1}{2}m} e^{-\frac{1}{2}r\sum_1^m \tau_i^2} dr \\
&= \frac{1}{(\sqrt{2\pi})^m} \frac{1}{(\sqrt{2})^n \Gamma(\frac{1}{2}n)} \int_0^\infty e^{-\frac{1}{2}r\left(1+\sum_1^m \tau_i^2\right)} r^{\frac{1}{2}(n+m-2)} dr \\
&= \frac{\left(1 + \sum_1^n \tau_i^2\right)^{-\frac{1}{2}(m+n)}}{(\sqrt{\pi})^m 2^{\frac{1}{2}(m+n)} \Gamma(\frac{1}{2}n)} \int_0^\infty e^{-\frac{1}{2}\zeta} \zeta^{\frac{1}{2}(n+m-2)} d\zeta \\
&= \frac{\Gamma(\frac{1}{2}(n+m))}{(\sqrt{\pi})^m \Gamma(\frac{1}{2}n)} \left(1 + \sum_1^m \tau_i^2\right)^{-\frac{1}{2}(m+n)}.
\end{aligned} \tag{27}$$

Then let

$$\eta' = \frac{\zeta}{\sqrt{r}} = \frac{y_1}{\sqrt{r}} = u_1, \qquad \tau'^2 = \frac{\chi^2}{r} = u_2^2 + \cdots + u_m^2. \tag{28}$$

The joint distribution of η' and τ'^2 is thus, by (27),

$$\begin{aligned}
&P\{\eta' < \eta, \tau'^2 < \tau^2\} \\
&= \frac{\Gamma(\frac{1}{2}(m+n))}{(\sqrt{\pi})^m \Gamma(\frac{1}{2}n)} \underset{u_1<\eta,\ \sum_2^m u_i^2<\tau^2}{\int\int\cdots\int} \left(1 + \sum_1^m u_i^2\right)^{-\frac{1}{2}(m+n)} du_1 \cdots du_m \\
&= \frac{\Gamma(\frac{1}{2}(m+n))}{(\sqrt{\pi})^m \Gamma(\frac{1}{2}n)} \underset{u_1<\eta,\ \sum_2^m y_i^2<\tau^2/(1+u_1^2)}{\int\int\cdots\int} (1 + u_1^2)^{-\frac{1}{2}(m+n)+\frac{1}{2}(m-1)} \\
&\qquad \cdot \left(1 + \sum_2^m y_i^2\right)^{-\frac{1}{2}(m+n)} du_1\, dy_2 \cdots dy_m \\
&= \frac{\Gamma(\frac{1}{2}(m+n))}{(\sqrt{\pi})^m \Gamma(\frac{1}{2}n)} \underset{u_1<\eta,\ \sum y_i^2<\tau^2/(1+u_1^2)}{\int\int\cdots\int} (1 + u_1^2)^{-\frac{1}{2}(n+1)} \\
&\qquad \cdot \left(1 + \sum_2^m y_i^2\right)^{-\frac{1}{2}(m+n)} du_1\, dy_2 \cdots dy_m.
\end{aligned} \tag{29}$$

In order to evaluate this integral, we use the fact that the distribution of a ratio of χ^2_{m-1} to χ^2_{n+1}, the two being independent, can be expressed in two forms, by (27) and Wilks [2], p. 114,

$$\begin{aligned}
P\{\chi^2_{m-1}/\chi^2_{n+1} < \psi\} &= \frac{\Gamma(\frac{1}{2}(m+n))}{\Gamma(\frac{1}{2}(m-1))\Gamma(\frac{1}{2}(n+1))} \int_0^\psi \varphi^{\frac{1}{2}(m-1)-1}(1+\varphi)^{-\frac{1}{2}(m+n)} d\varphi \\
&= \frac{\Gamma(\frac{1}{2}(m+n))}{(\sqrt{\pi})^{m-1}\Gamma(\frac{1}{2}(n+1))} \underset{\sum_1^{m-1} q_i^2<\psi}{\int\cdots\int} \left(1 + \sum_1^{m-1} q_i^2\right)^{-\frac{1}{2}(m+n)} dq_1 \cdots dq_m,
\end{aligned} \tag{30}$$

so that

$$P\{\eta' < \eta, \tau'^2 < \tau^2\}$$

$$= \frac{\Gamma(\frac{1}{2}(m+n))}{\sqrt{\pi}\,\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(m-1))} \times \int_{u_1<\eta} \int_{\varphi=0}^{\varphi=\tau^2/(1+u_1^2)} (1+u_1^2)^{-\frac{1}{2}(n+1)} \varphi^{\frac{1}{2}(m-3)} (1+\varphi)^{-\frac{1}{2}(m+n)}\, d\varphi\, du_1$$

$$(31) \quad = \frac{\Gamma(\frac{1}{2}(m+n))}{\sqrt{\pi}\,\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(m-1))} \times \int_{u=-\infty}^{\eta} \int_{\zeta=0}^{\tau^2} (1+u^2)^{-\frac{1}{2}(n+1)} \zeta^{\frac{1}{2}(m-3)} \left(1 + \frac{\zeta}{1+u^2}\right)(1+u^2)^{-\frac{1}{2}(m-3)-1}\, d\zeta\, du$$

$$= \frac{\Gamma(\frac{1}{2}(m+n))}{\sqrt{\pi}\,\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(m-1))} \int_{u_1=-\infty}^{\eta} \int_{\zeta=0}^{\tau^2} \zeta^{\frac{1}{2}(m-3)} (1+u^2+\zeta)^{-\frac{1}{2}(m+n)}\, d\zeta\, du.$$

Now we wish to find the distribution of

$$F' = \sum_1^m (t_i - c_i)^2$$

$$(32) \quad = \frac{\sum_1^m (x_i - c_i\sqrt{r})^2}{r} = \frac{\chi^2}{r} + \frac{(\zeta - \sqrt{r}\sqrt{\Sigma c_i^2})^2}{r}$$

$$= \tau'^2 + (\eta' - \sqrt{\Sigma c_i^2})^2.$$

Carrying out the transformation (32), it is found that the joint density function of η' and F' is

$$p(\eta', F')\, d\eta'\, dF'$$

$$= \frac{\Gamma(\frac{1}{2}(m+n))}{\sqrt{\pi}\,\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(m-1))} [F' - (\eta' - \sqrt{\Sigma c_i^2})^2]^{\frac{1}{2}(m-3)}$$

$$(33) \quad \times [1 + \eta'^2 + F' - (\eta' - \sqrt{\Sigma c_i^2})^2]^{-\frac{1}{2}(m+n)}\, d\eta'\, dF'$$

$$= \frac{\Gamma(\frac{1}{2}(m+n))}{\sqrt{\pi}\,\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(m-1))} [F' - \rho^2]^{\frac{1}{2}(m-3)}$$

$$\times [1 + F' + 2\rho\sqrt{\Sigma c_i^2} + \Sigma c_i^2]^{-\frac{1}{2}(m+n)}\, d\rho\, dF',$$

where $\rho = \eta' - \sqrt{\Sigma c_i^2}$. In order to obtain the distribution of F' we must integrate out ρ over $-\sqrt{F} < \rho < \sqrt{F}$, obtaining

$$P\{F' < T\} = \Phi_{m,n}(T, \Sigma c_i^2)$$

$$(34) \quad = \frac{\Gamma(\frac{1}{2}(m+n))}{\sqrt{\pi}\,\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(m-1))} \times \int_{F'=0}^{T} \int_{\rho=-\sqrt{F'}}^{\sqrt{F'}} [F' - \rho^2]^{\frac{1}{2}(m-3)} [1 + F' + 2\rho\sqrt{\Sigma c_i^2} + \Sigma c_i^2]^{-\frac{1}{2}(m+n)}\, d\rho\, dF'.$$

In the case $\Sigma c_i^2 = 0$, (34) reduces to the distribution of the ratio χ_m^2/χ_n^2.

4. Test of a linear hypothesis. In this case the power of the test usually employed is affected not only by the variance, but also by the values of the predictors. In order to avoid this difficulty, it will be assumed that only a predetermined number of different sets of predictors are used, and that these sets are repeated as a whole, as many times as is necessary. This covers, in particular, the replication of orthogonal designs for the analysis of variance.

Let y_{ij}, $i = 1 \cdots m$, $j = 1, 2, \cdots$ be independently normally distributed with means

$$Ey_{ij} = \sum_{k=1}^{\mu} a_k x_{ki}, \qquad \mu \le m, \qquad \text{rank } (x_{ki}) = \mu, \tag{35}$$

and variance σ^2, the x_{ki} being given in advance, σ^2 and a_k unknown. We wish to test H_0: $\sum_{k=1}^{\mu} c_{lk} a_k = c_{l0}$, $l = 1 \cdots r \le \mu$, where we may suppose equations (36) linearly independent, the c_{lk} being given constants. It will be convenient to reduce this to a canonical form, as in Tang [3]. First, by a non-singular linear transformation

$$x_{ki} = \sum_{l=1}^{\mu} b_{kl} z_{li} \tag{37}$$

we can make

$$\sum_{i=1}^{m} \begin{pmatrix} z_{1i} \\ \cdot \\ \cdot \\ \cdot \\ z_{\mu i} \end{pmatrix} (z_{1i} \cdots z_{\mu i}) = I_\mu, \qquad \text{the } \mu \times \mu \text{ identity matrix,} \tag{38}$$

any two sets of b_{kl} that accomplish this being related by an orthogonal transformation. Then (35) becomes

$$\begin{aligned} Ey_{ij} &= \sum_{k=1}^{\mu} a_k \sum_{j=1}^{\mu} b_{kl} z_{li} \\ &= \sum_{l=1}^{\mu} \left(\sum_{k=1}^{\mu} a_k b_{kl} \right) z_{li} = \sum_{k=1}^{\mu} a'_k z_{ki}, \end{aligned} \tag{39}$$

and (34) becomes

$$\begin{aligned} c_{l0} &= \sum_{k=1}^{\mu} c_{lk} a_k = \sum_{k=1}^{\mu} c_{lk} \sum_{m=1}^{\mu} a'_m b^{mk} \\ &= \sum_{m=1}^{\mu} a'_m \sum_{k=1}^{\mu} c_{lk} b^{mk} \\ &= \sum_{m=1}^{\mu} c'_{lm} a'_m, \qquad l = 1 \cdots r \le \mu, \end{aligned} \tag{40}$$

where b^{mk} are such that $\Sigma b^{mk} b_{kl} = \delta_{ml}$, the Kronecker delta, or, in matrix notation $(b_{km})^{-1} = (b^{km})$. Next, the equations (40) can be made into an orthonormal set

$$c''_{l0} = \sum_{m=1}^{\mu} c_{lm} a'_m \tag{41}$$

i.e., one in which

$$\sum_{m=1}^{\mu} c''_{km} c''_{lm} = \delta_{kl} \tag{42}$$

by a non-singular linear transformation on the c'_{lm}. Clearly $\Sigma c''^2_{l0}$ is an invariant of (41), i.e., it does not depend upon the choice of a particular transformation (37), or of a particular transformation of the c'_{lm} into c''_{lm}, since, in both cases, all admissible transformations are connected by an orthogonal transformation. Then we define

$$y'_{ij} = \sum_{q=1}^{m} z_{iq} y_{qj}, \qquad i = 1, \cdots, \mu \tag{43}$$

$$y'_{ij} = \sum_{q=1}^{m} d_{iq} y_{qj}, \qquad i = \mu + 1, \cdots, m \tag{44}$$

in such a way that $\begin{pmatrix} z_{iq} \\ d_{iq} \end{pmatrix}$ is an orthogonal matrix which is possible, by **(38)**. Then

$$\begin{aligned} Ey'_{ij} &= \sum_{q=1}^{m} z_{iq} E y_{qj} = \sum_{q=1}^{m} z_{iq} \sum_{k=1}^{\mu} z_{kq} a'_k \\ &= \sum_{k=1}^{\mu} a'_k \sum_{q=1}^{m} z_{iq} z_{kq} = a'_i \quad \text{for} \quad i = 1, \cdots, \mu, \end{aligned} \tag{45}$$

$$\begin{aligned} Ey'_{ij} &= \sum_{q=1}^{m} d_{iq} E y_{qj} = \sum_{q=1}^{m} d_{iq} \sum_{k=1}^{\mu} z_{kq} a'_k \\ &= \sum_{k=1}^{\mu} a'_k \sum_{q=1}^{m} d_{iq} z_{kq} = 0 \quad \text{for} \quad i = \mu + 1, \cdots, m. \end{aligned} \tag{46}$$

Finally we define

$$y''_{ij} = y'_{ij}, \qquad i = \mu + 1 \cdots, m \tag{47}$$

$$y''_{ij} = \sum_{m=1}^{\mu} c_{im} y'_{mj}, \qquad i = 1, \cdots, r \tag{48}$$

$$y''_{ij} = \sum_{m=1}^{\mu} e_{im} y'_{mj}, \qquad i = r + 1, \cdots, \mu, \tag{49}$$

where the e_{im} are such that $\begin{pmatrix} c_{im} \\ e_{im} \end{pmatrix}$ is an orthogonal matrix. Since the transformation applied to the y_{ij} to obtain y''_{ij} is orthogonal, the y''_{ij} are independently normally distributed with variance σ^2. Also

$$Ey''_{ij} = 0, i = \mu + 1, \cdots, t \tag{50}$$

$$Ey''_{ij} = \sum_{m=1}^{\mu} c_{im} a'_m = c_{i0}, \qquad i = 1, \cdots, r \tag{51}$$

$$Ey''_{ij} = \sum_{m=1}^{\mu} e_{im} a'_m, \qquad i = r + 1, \cdots, \mu. \tag{52}$$

Since (50), (51), (52) were obtained from the original formulation by a non-singular linear transformation, the derivation can be reversed, which implies the equivalence of (50), (51), (52) to the problem as originally formulated.

Thus we can restate the problem in the following manner. Let y_{ij}, $i = 1, \cdots, t$, $j = 1, 2, \cdots$ be independently normally distributed with variance σ^2 and means

$$\begin{aligned} Ey_{ij} &= \xi_i, i = 1, \cdots, \mu \\ Ey_{ij} &= 0, \quad i = \mu + 1, \cdots, t, \ \xi_i \text{ and } \sigma^2 \text{ unknown.} \end{aligned} \tag{53}$$

We wish to test

$$H_0 : \xi_i = 0, \ i = 1, \cdots, p \leq \mu \tag{54}$$

the ξ_i for $i = p + 1 \cdots \mu$ and σ^2 being nuisance parameters.

Obtain a first sample y_{ij}, $i = 1, \cdots, t$, $j = 1, \cdots, n_0$. Estimate the variance by

$$s^2 = \frac{1}{n_0 t - \mu} \left\{ \sum_{j=1}^{n_0} \sum_{i=1}^{t} y_{ij}^2 - \frac{1}{n_0} \sum_{i=1}^{\mu} \left(\sum_{j=1}^{n_0} y_{ij} \right)^2 \right\}. \tag{55}$$

Let z be a predetermined constant, and n be defined by

$$n = \max \left\{ \left[\frac{s^2}{z} \right] + 1, n_0 + 1 \right\}. \tag{56}$$

After s^2 has been obtained, determine a set of real numbers, $a_1 \cdots a_n$, in accordance with a preassigned rule, so as to satisfy

$$\begin{aligned} \Sigma a_j &= 1 \\ s^2 \Sigma a_j^2 &= z \\ a_1 = \cdots &= a_{n_0}. \end{aligned} \tag{57}$$

Then

$$F' = \frac{\sum_{i=1}^{p} \left(\sum_{j=1}^{n} a_j y_{ij} \right)^2}{z(n_0 t - \mu)} \tag{58}$$

has the non-central F-distribution given by (34) with $n = n_0 t - \mu$, $m = p$ and

$$\sum_1^p c_i^2 = \sum_1^p \xi_i^2 / (n_0 t - \mu) z, \tag{59}$$

where ξ_i are the true means, allowing for the possibility that H_0 is not true. For, $(n_0 t - \mu)s^2/\sigma^2$ has the distribution of $\chi^2_{n_0 t - \mu}$, and, after it has been determined, $\sum_{j=1}^{n} a_j y_{ij} - \xi_i$, $i = 1 \cdots r$, are independently normally distributed with mean 0 and variance $\sigma^2 \Sigma a_i^2 = \sigma^2 z / s^2$, so that, given s^2, $\left(\sum_{j=1}^{n} a_j y_{ij} - \xi_i \right) / \sqrt{z}$, $i = 1 \cdots p$

are independently normally distributed with mean 0 and variance σ^2/s^2. But the random variables t_i, in section 3 are of the form $x_i/\sqrt{r}$ where the x_i are independently normally distributed with mean 0 and variance σ^2, while r/σ^2 has the $\chi^2_{n_0 t-\mu}$ distribution independent of the x_i. Thus t_i can be considered to have been obtained by first selecting a stochastic variable r such that r/σ^2 has the distribution of $\chi^2_{n_0 t-\mu}$ and then selecting t_i to be independently normally distributed, given r, with mean 0 and variance σ^2/r. Since r corresponds with $(n_0 t - \mu)s^2$, comparing this with the above, we find that

$$(60) \qquad \frac{\sum_{j=1}^{n} a_j y_{ij} - \xi_i}{\sqrt{z}\sqrt{n_0 t - \mu}}, \qquad i = 1 \cdots p$$

have the same joint distribution as the t_i. The $\frac{\xi_i}{\sqrt{(n_0 t - s)z}}$ are constants, so that

$$(61) \qquad F' = \frac{\sum_{i=1}^{p}\left(\sum_{j=1}^{n} a_j y_{ij}\right)^2}{z(n_0 t - \mu)} = \sum_{i=1}^{p}\left\{\frac{\sum_{j=1}^{n} a_j y_{ij} - \xi_i}{\sqrt{z(n_0 t - \mu)}} + \frac{\xi_i}{\sqrt{z(n_0 t - \mu)}}\right\}^2$$

has the same distribution (34) as $\sum_{i=1}^{p} (t_i - c_i)^2$ with $c_i = \xi_i/\sqrt{(n_0 t - \mu)z}$.

The tests of significance and confidence regions are obtained by a procedure completely analogous to that used in the case of Student's hypothesis. If we define $k = F_{p,n_0 t-\mu,\alpha}$ by

$$(62) \qquad P\{F_{p,n_0 t-\mu} > k\} = \alpha,$$

then a critical region of size α for testing H_0 is given by

$$(63) \qquad \frac{n_0 t - \mu}{p} F' > k.$$

Its power function is

$$(64) \qquad 1 - \beta(\xi) = 1 - \Phi_{p,n_0 t-\mu}\left(k, \frac{\sum_{1}^{p} \xi_i^2}{z(n_0 t - \mu)}\right).$$

Similarly, a confidence region for ξ_i, $i = 1 \cdots p$, of confidence coefficient $1 - \alpha$ is given by the set of all ξ_i such that

$$(65) \qquad \frac{n_0 t - \mu}{p} F'(\xi_1 \cdots \xi_p) < k,$$

where

$$(66) \qquad F'(\xi_1 \cdots \xi_p) = \frac{\sum_{i=1}^{p}\left(\sum_{j=1}^{n} a_j y_{ij} - \xi_i\right)^2}{z(n_0 t - \mu)}.$$

It is evident that this defines the interior of the hypersphere

$$\sum_{i=1}^{p}\left(\xi_i - \sum_{j=1}^{n} a_j y_{ij}\right)^2 < kzp \tag{67}$$

whose volume is independent of the variance σ^2.

The distribution of n, the required number of sets of observations for the above tests and confidence intervals is given by

$$\begin{aligned} P\{n = n_0 + 1\} &= P\left\{\frac{s^2}{z} \le n_0 + 1\right\} \\ &= P\{(n_0 t - \mu)s^2/\sigma^2 < (n_0 + 1)(n_0 t - \mu)z/\sigma^2\} \\ &= P\{\chi_\delta^2 < y\} = \frac{1}{(\sqrt{2})^\delta\,\Gamma(\frac{1}{2}\delta)}\int_0^y e^{-\frac{1}{2}u}u^{\frac{1}{2}\delta-1}\,du, \end{aligned} \tag{68}$$

where

$$\begin{aligned} y &= (n_0 + 1)(n_0 t - \mu)z/\sigma^2 \\ \delta &= n_0 t - \mu \end{aligned} \tag{69}$$

and

$$\begin{aligned} P\{n = \nu\} &= P\left\{\nu < \frac{s^2}{z} + 1 < \nu + 1\right\} \\ &= P\{(\nu - 1)\delta z/\sigma^2 < \chi_\delta^2 < \nu\delta z/\sigma^2\} \\ &= \frac{1}{(\sqrt{2})^\delta\,\Gamma(\frac{1}{2}\delta)}\int_{(\nu-1)\delta z/\sigma^2}^{\nu\delta z/\sigma^2} e^{-\frac{1}{2}u}u^{\frac{1}{2}\delta-1}\,du, \end{aligned} \tag{70}$$

for integral $\nu > n_0 + 1$, all other values being impossible.
Thus $E(n)$ satisfies the inequalities

$$\begin{aligned} &\frac{1}{(\sqrt{2})^\delta\,\Gamma(\frac{1}{2}\delta)}\left\{\int_0^y (n_0 + 1)e^{-\frac{1}{2}u}u^{\frac{1}{2}\delta-1}\,du + \int_y^\infty e^{-\frac{1}{2}u}u^{\frac{1}{2}\delta-1}\frac{\sigma^2 u}{\delta z}\,du\right\} \\ &< E(n) \\ &< \frac{1}{(\sqrt{2})^\delta\,\Gamma(\frac{1}{2}\delta)}\left\{\int_0^y (n_0 + 1)e^{-\frac{1}{2}u}u^{\frac{1}{2}\delta-1}\,du + \int_y^\infty e^{-\frac{1}{2}u}u^{\frac{1}{2}\delta-1}\left(\frac{\sigma^2 u}{\delta z} + 1\right)du\right\}, \end{aligned} \tag{71}$$

which can be rewritten

$$\begin{aligned} &(n_0 + 1)P\{\chi_\delta^2 < y\} + \frac{\sigma^2}{z}P\{\chi_{\delta+2}^2 > y\} \\ &< E(n) \\ &< (n_0 + 1)P\{\chi_\delta^2 < y\} + \frac{\sigma^2}{z}P\{\chi_{\delta+2}^2 > y\} + P\{\chi_\delta^2 > y\}. \end{aligned} \tag{72}$$

The modifications required to avoid wasting information are exactly analogous to those made in the case of the test for Student's hypothesis.

5. Non existence of a single-sample test for a linear hypothesis whose power is independent of the variance. The canonical form (see Tang [3]) for a linear hypothesis in the single sample case can be derived immediately from (53) and (54). Let x_i, $i = 1 \cdots n$ be independently normally distributed with means

$$Ex_i = \xi_i, \; i = 1 \cdots p \tag{73}$$

$$Ex_i = 0, \; i = p + 1 \cdots n$$

and variance σ^2. The ξ_i and σ^2 are unknown, and we wish to test $H_0 : \xi_i = 0$, $i = 1 \cdots p$.

The most powerful test for H_0 against a given alternative $\xi_i = \xi_{i0}$, $i = 1 \cdots p$, if the variance σ^2 is known, is that based upon the probability ratio (see Neyman and Pearson [4])

$$\frac{p_1}{p_0} = \frac{\dfrac{1}{(\sqrt{2\pi}\,\sigma)^n} e^{-\frac{1}{2\sigma^2}\left\{\sum_1^p (x_i - \xi_{i0})^2 + \sum_{p+1}^n x_i^2\right\}}}{\dfrac{1}{(\sqrt{2\pi}\,\sigma)^n} e^{-\frac{1}{2\sigma^2}\sum_1^n x_i^2}} = e^{-\frac{1}{2\sigma^2}\left\{\sum_1^p \xi_{i0}^2 - 2\sum_1^p \xi_{i0} x_i\right\}}. \tag{74}$$

Since any strictly increasing function of p_1/p_0 is equivalent for this purpose, we can use

$$\varphi(x_1 \cdots x_p) = \sum_1^p \xi_{i0} x_i . \tag{75}$$

The critical region of size α based upon φ is given by

$$W_0(\sigma) = E\left\{x \,\middle|\, \frac{\sum_1^p \xi_{i0} x_i}{\sigma \sqrt{\sum_1^p \xi_{i0}^2}} > z\right\}, \tag{76}$$

where

$$\frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-\frac{1}{2}x^2}\,dx = \alpha, \tag{77}$$

since, under H_0, $\sum_1^p \xi_{i0} x_i$ is normally distributed with mean 0 and variance $\sigma^2 \sum_1^p \xi_{i0}^2$. Under H_1, $\sum_1^p \xi_{i0} x_i$ is normally distributed with mean $\sum_1^p \xi_{i0}^2$ and variance $\sigma^2 \sum_1^p \xi_{i0}^2$. Thus the power of the test for the alternative H_1 as a function of σ^2 is

$$
\begin{aligned}
1 - \beta_0(\sigma) &= P\{x \in W_0(\sigma) \mid \xi_i = \xi_{i0}, \sigma^2\} \\
&= P\left\{\frac{\sum_1^p \xi_{i0} x_i - \sum_1^p \xi_{i0}^2}{\sigma\sqrt{\sum_1^p \xi_{i0}^2}} > z - \frac{\sqrt{\sum_1^p \xi_{i0}^2}}{\sigma}\right\} \\
&= \frac{1}{\sqrt{2\pi}} \int_{z - \frac{\sqrt{\Sigma\xi_{i0}^2}}{\sigma}}^{\infty} e^{-\frac{1}{2}x^2}\, dx.
\end{aligned} \tag{78}
$$

Now let us suppose there exists a test based on the critical region W of size α whose power $1 - \beta$ is independent of σ^2. Since $W_0(\sigma)$ is the best critical region of size α for any σ we must have

$$
1 - \beta \leq 1 - \beta_0(\sigma) = \frac{1}{\sqrt{2\pi}} \int_{z - \frac{\sqrt{\Sigma\xi_{i0}^2}}{\sigma}}^{\infty} e^{-\frac{1}{2}x^2}\, dx, \tag{79}
$$

so that

$$
1 - \beta \leq \underset{\sigma}{\text{g.l.b.}}\, [1 - \beta_0(\sigma)] = \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \alpha. \tag{80}
$$

By interchanging H_0 and H_1 we can reverse the inequality (80), proving

$$
1 - \beta = \alpha. \tag{81}
$$

Thus any single-sample test for a linear hypothesis whose power is independent of the variance has constant power equal to the size of the critical region.

REFERENCES

[1] GEORGE B. DANTZIG, "On the non-existence of tests of "Student's" hypothesis having power functions independent of σ," *Annals of Math. Stat.*, Vol. 11 (1940), p. 186.

[2] S. S. WILKS, *Mathematical Statistics*, Princeton, 1943.

[3] P. C. TANG, "The power function of the analysis of variance tests," *Stat. Res. Mem.*, Vol. 2 (1938).

[4] NEYMAN AND PEARSON, *Stat. Res. Mem.*, Vol. 1 (1936).

[5] A. WALD, "Sequential tests of statistical hypotheses," *Annals of Math. Stat.*, Vol. 16, June 1945.