This is a general result, applicable to any arrangement of the terms of an arbitrary square matrix A, subject only to the conditions that $|A| \neq 0$ and that no diagonal term of A is zero. In this latter exceptional case, the iterative method itself obviously cannot be applied.

The criterion (11) clearly shows that the order in which the elements of the matrix A are arranged is important. For instance, it is plain that an arrangement in which the diagonal terms are large and the off-diagonal terms, particularly the post-diagonal terms, are small will tend to favor convergence.

A somewhat relaxed condition, which is sufficient but not necessary, is obtained through the use of an inequality used by Hotelling[3], namely,

$$N(B^m) \leq [N(B)]^m, \tag{12}$$

in which $N(B)$ is the norm of the matrix B, that is, the square root of the sum of the products of its elements by their complex conjugates, or in the case of a real matrix the square root of the sum of the squares of the elements.

The condition is that, if

$$N(A_1^{-1}A_2) < 1, \tag{13}$$

then

$$\lim_{m\to\infty} (A_1^{-1}A_2)^m = 0. \tag{14}$$

Criterion (13) is readily computed, since A_1^{-1}, the reciprocal of a triangular matrix is readily computed, and the post-multiplication by A_2 involves a number of zero terms.

A more stringent condition than (13) though still not a necessary condition, is that if some finite number p can be found such that

$$N(A_1^{-1}A_2)^p < 1, \tag{15}$$

then (14) follows. Since n matrix squarings result in a value of $p = 2^n$, the size of the norm for fairly large values of p can be investigated without excessive labor.

A REMARK ON INDEPENDENCE OF LINEAR AND QUADRATIC FORMS INVOLVING INDEPENDENT GAUSSIAN VARIABLES

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The purpose of this note is to call attention to the following useful theorem, which to the best of my knowledge was never stated explicitly.

If $X_1, X_2, X_3, \cdots X_n$ are identically distributed, independent Gaussian random variables each having mean 0, *then the necessary and sufficient condition that*

$$\sum_{j,k=1}^{n} a_{jk} X_j X_k \quad \textit{and} \quad \sum_{j=1}^{n} \alpha_j X_j = \alpha \cdot X$$

be independent, is that

$$A\alpha = O,$$

where A *is the matrix of the quadratic form,* α *the vector* $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ *and* X *the vector* $(X_1, X_2, \cdots X_n)$.

PROOF OF SUFFICIENCY.[1] Since $A\alpha = O$, it follows that 0 is an eigenvalue of A, and α is a corresponding eigenvector.

Denoting by $\lambda_2, \cdots, \lambda_n$ the remaining eigenvalues and by $\beta_2, \cdots, \beta_n$ the corresponding eigenvectors, we have

$$\sum_{j,k=1}^{n} a_{jk} X_j X_k = \sum_{j=2}^{n} \lambda_j (\beta_j \cdot X)^2.$$

Since the β's are orthogonal to α, it follows that the linear combinations $\beta_j \cdot X$ are independent of $\alpha \cdot X$, and this completes the proof.

PROOF OF NECESSITY. From the assumption of independence it follows that

$$\sum_{j,k=1}^{n} a_{jk} X_j X_k \quad \textit{and} \quad \left(\sum_{j=1}^{n} \alpha_j X_j\right)^2 = \sum_{j,k=1}^{n} \alpha_j \alpha_k X_j X_k$$

are independent. Thus by Craig's theorem[2]

$$AB = O$$

where $B = ((\alpha_j \alpha_k))$.

This implies almost immediately that $A\alpha = O$.

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[1] *Added in proof:* Dr. L. Guttman has kindly pointed out to me that the proof of sufficiency given here has been used by D. Jackson in the article "Mathematical principles in the theory of small samples", *Amer. Math. Month.*, Vol. 42 (1935), pp. 344–364, see in particular pp. 354–355. Jackson considers only the independence of $\bar{x}$ and s^2, which is of crucial importance in deriving student's distribution.

[2] A. T. CRAIG, *Annals of Math. Stat.*, Vol. 14 (1943), pp. 195–197; see also H. HOTELLING, ibid., Vol. 15 (1944), pp. 427–429.