THE VARIANCE OF THE MEASURE OF A TWO-DIMENSIONAL RANDOM SET

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1. Introduction. In a recent paper H. E. Robbins[1] has solved the problem of the variance of the measure of a one-dimensional random set. The present paper treats a similar problem relating to a two-dimensional random set under somewhat more general conditions.

Let R denote a rectangle of dimensions $a \times b$ whose position is fixed. Let R' denote another fixed rectangle concentric with R, its sides $a + \gamma$ and $b + \gamma$ (where $\gamma > 0$) being parallel to the sides a and b respectively of R. Finally, let ρ denote a rectangle of fixed dimensions but variable position, whose sides $\alpha < 2\gamma$ and $\beta < 2\gamma$ are parallel to a and b respectively, but the position of whose center will be considered as random. In fact it will be assumed that the rectangle ρ is dropped on the plane of R in a manner which satisfies the following two assumptions:

(i) The probability that the center of ρ falls within R' exactly s times has a defined value P_s for each $s = 0, 1, 2, \cdots$ Thus, if $\Psi(u)$ denotes the probability generating function of s, so that

$$\Psi(u) = \sum_{s=0}^{\infty} u^s P_s, \tag{1}$$

then $\Psi(u)$ is assumed known but will be left arbitrary till the general result is obtained.

(ii) Whenever a fixed number s of centers of ρ fall within R', it will be assumed that the probability that exactly k centers of ρ fall within any chosen sub-area w contained in R' is given by the binomial expression

$$\frac{s!}{k!(s-k)!} \frac{w^k}{R'} \left(1 - \frac{w}{R'}\right)^{s-k} \tag{2}$$

Under the above conditions, denote by E the set of all those points of R which are covered at least once by the rectangle ρ during the course of the trials considered. Let X denote the measure of E. The purpose of this paper is to evaluate the first two moments of X.

First, the computations will be made for the case when s is fixed, i.e. when

$$\Psi(u) = u^s. \tag{3}$$

The values of the two moments of X computed for fixed s will be denoted by $M_1(a,b|s)$ and $M_2(a,b|s)$. Next, the moments of X will be evaluated for an arbitrary generating function $\Psi(u)$, and these will be denoted by $M_1(a, b)$ and $M_2(a, b)$.

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[1] H. E. Robbins, "On the measure of a random set", *Annals of Math. Stat.*, Vol. 15 (1944), pp. 70–74.

H. E. Robbins has found the first moment

$$M_1(a, b \mid s) = ab\left\{1 - \left(1 - \frac{\alpha\beta}{R'}\right)^s\right\} \tag{4}$$

Also, for a one-dimensional set, he has obtained the second moment, say $M_2(a|s)$, when $\alpha \leq a$.

It follows immediately from (4) and (1) that, whatever be the probability generating function $\Psi(u)$,

$$M_1(a, b) = ab\left\{1 - \Psi\left(1 - \frac{\alpha\beta}{R'}\right)\right\} \tag{5}$$

In particular, if the probabilities P_s are those of Poisson when the density of positions of the center of ρ per unit of area is λ, so that

$$\Psi(u) = e^{\lambda R'(u-1)}, \tag{6}$$

then

$$M_1(a,b) = ab\{1 - e^{-\alpha\beta\lambda}\} \tag{7}$$

Our remaining problem, therefore, is that of evaluating the second moment of X. Instead we shall evaluate the second moment of

$$Y = ab - X, \tag{8}$$

and shall denote it by $m(a, b \mid s)$ or $m(a, b)$ according as s is or is not considered to be fixed.

2. Derivative of the second moment of Y. In order to evaluate $m(a, b)$, we begin by calculating its second (mixed) derivative, say $D(a, b \mid s)$, where

$$\begin{aligned} D(a, b \mid s) &= \frac{\partial^2 m(a, b \mid s)}{\partial a \partial b} \\ &= \lim_{\Delta a, \Delta b \to 0} \frac{1}{\Delta a \Delta b} \{m(a + \Delta a, b + \Delta b \mid s) - m(a, b + \Delta b \mid s) \\ &\qquad - m(a + \Delta a, b \mid s) + m(a, b \mid s)\} \\ &= \lim \frac{1}{\Delta a \Delta b} I(\Delta a, \Delta b) \quad \text{(say)}, \end{aligned} \tag{9}$$

where Δa and Δb are the increments of a and b respectively. Once $D(a, b \mid s)$ is found, the formula for $m(a, b \mid s)$ will be obtained by two quadratures. For definiteness we shall assume Δa and Δb both to be positive, but of course the argument which follows applies equally to other cases.

Consider the rectangle of dimensions $(a + \Delta a)$ and $(b + \Delta b)$ as shown in Figure 1, and denote by U, V and W the measures of the "uncovered" parts of the three rectangles $\Delta a \times b$, $a \times \Delta b$, and $\Delta a \times \Delta b$ respectively. That is to say, U, V and W are defined with respect to these three rectangles precisely in the same manner in which Y is defined with respect to the original rectangle $a \times b \equiv R$. Using the letter E to denote the expectation, we easily find that

$$I(\Delta a, \Delta b) = 2E(YW) + 2E(UV) + 2E(VW) + 2E(UW) + E(W^2). \tag{10}$$

However, each of the three expectations in the second line of formula (10) is infinitesimal of an order higher than the product $\Delta a \Delta b$. In fact, none of the

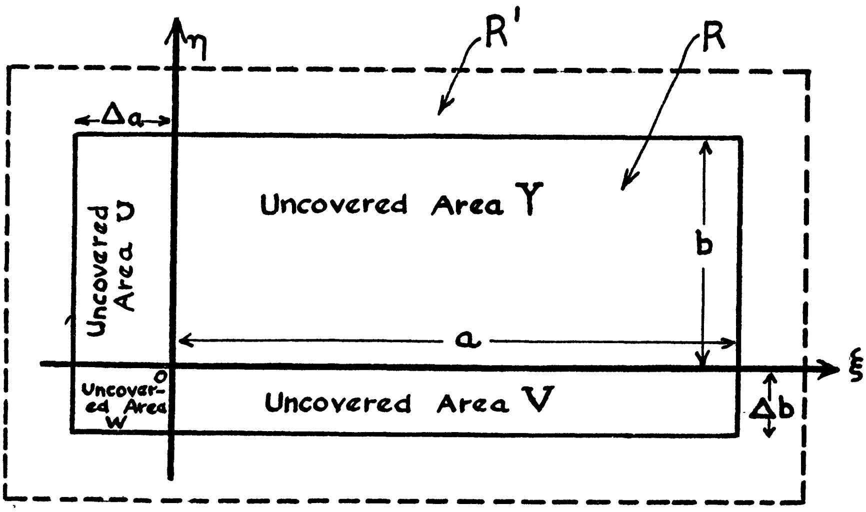

FIGURE 1.

variables U, V and W can exceed the area of the rectangle of which it forms part; that is,

$$\begin{aligned} 0 &\leq U \leq b\Delta a, \\ 0 &\leq V \leq a\Delta b, \\ 0 &\leq W \leq \Delta a\Delta b. \end{aligned} \tag{11}$$

It follows that

$$\begin{aligned} 0 &\leq E(UW) \leq b(\Delta a)^2\Delta b, \\ 0 &\leq E(VW) \leq a\Delta a(\Delta b)^2, \\ 0 &\leq E(W^2) \leq (\Delta a\Delta b)^2. \end{aligned} \tag{12}$$

Hence, from (9), (10) and (12)

$$D(a, b \mid s) = 2 \lim \frac{1}{\Delta a \Delta b} \{E(YW) + E(UV)\}. \tag{13}$$

We now reduce the calculation of (13) to finite form by approximating to the infinite sets Y, U, V, W by progressively more ample but finite sets. To do so, we cover R' by progressively more ample but finite networks of points. More precisely: consider a rectangular system of axes $O\xi$ and $O\eta$ oriented as in Figure 1 so that the axes are common boundaries of $a \times b \equiv R$ and of the rectangles obtained by increasing a and b. Let

$$d_n = a/(n + 1), \qquad \delta_n = b/(n + 1). \tag{14}$$

Consider the lattice of points (ij) with coordinates

$$\xi_i^{(n)} = id_n, \qquad \eta_j^{(n)} = j\delta_n \tag{15}$$

for $i = -v_1^{(n)}, -v_1^{(n)} + 1, \cdots, 0, 1, 2, \cdots, n; j = -v_2^{(n)}, -v_2^{(n)} + 1, \cdots, 0, 1, 2, \cdots, n$, where $v_1^{(n)}$ and $v_2^{(n)}$ are the greatest integers such that

$$v_1^{(n)} d_n \leq \Delta a \tag{16}$$

and

$$v_2^{(n)} \delta_n \leq \Delta b. \tag{17}$$

To simplify the writing, the superscripts (n) will henceforth be dropped.

With every point (ij) we associate a random variable x_{ij} defined as follows. If in the course of the trials contemplated none of the rectangles ρ covers (ij), then $x_{ij} = 1$. Otherwise $x_{ij} = 0$. Further, write

$$\begin{aligned} Y_n &= d_n \delta_n \sum_{i=0}^{n} \sum_{j=0}^{n} x_{ij}, \\ U_n &= d_n \delta_n \sum_{i=-v_1}^{0} \sum_{j=0}^{n} x_{ij}, \\ V_n &= d_n \delta_n \sum_{i=0}^{n} \sum_{j=-v_2}^{0} x_{ij}, \\ W_n &= d_n \delta_n \sum_{i=-v_1}^{0} \sum_{j=-v_2}^{0} x_{ij}. \end{aligned} \tag{18}$$

Now the boundary of the set E, for a fixed s, consists of one or more polygons having a finite total number of sides each of bounded length. It follows that, given any $\epsilon > 0$, there exists, for a fixed s, a number $N_\epsilon(s)$ such that $n > N_\epsilon(s)$ implies that

$$| Y_n - Y | < \epsilon, \tag{19}$$

with similar inequalities relating to U_n, V_n and W_n. Hence it follows immediately that

$$\begin{aligned} \lim_{n\to\infty} E(Y_n W_n \mid s) &= E(YW \mid s), \\ \lim_{n\to\infty} E(U_n V_n \mid s) &= E(UV \mid s). \end{aligned} \tag{20}$$

The expectations in formula (13) will therefore be obtained as limits of those on the left hand sides of (20). We have

$$E(Y_n W_n \mid s) = d_n^2 \delta_n^2 \sum_{i=-v_1}^{0} \sum_{j=-v_2}^{0} E\left(x_{ij} \sum_{k=0}^{n} \sum_{l=0}^{n} x_{kl} \mid s\right), \tag{21}$$

$$E(U_n V_n \mid s) = d_n^2 \delta_n^2 \sum_{i=-v_1}^{0} \sum_{j=0}^{n} E\left(x_{ij} \sum_{k=0}^{n} \sum_{l=-v_2}^{0} x_{kl} \mid s\right). \tag{22}$$

Hitherto we have made no assumptions concerning the values of Δa and Δb. Since these are to tend to zero, we may assume that

$$\begin{aligned} 0 &< \Delta a < \gamma - \alpha/2, \\ 0 &< \Delta b < \gamma - \beta/2. \end{aligned} \tag{23}$$

On this assumption, we shall now compute the expectations of the type $E(x_{ij}x_{kl} \mid s)$, of which (21) and (22) are linear combinations.

Since the variables x_{ij} and x_{kl} are capable only of the two values unity and zero, the expectation of their product is simply the probability that both of them are equal to unity, i.e. the probability that both points (ij) and (kl) are "missed" by all the s rectangles ρ falling on R'. This probability may have one of two forms. If both

$$d_n \mid i - k \mid < \alpha \quad \text{and} \quad \delta_n \mid j - l \mid < \beta, \tag{24}$$

then

$$E(x_{ij}x_{kl} \mid s) = \left\{1 - \frac{2\alpha\beta - (\alpha - d_n \mid i - k \mid)(\beta - \delta_n \mid j - l \mid)}{R'}\right\}^s; \tag{25}$$

while otherwise

$$E(x_{ij}x_{kl} \mid s) = \left(1 - \frac{2\alpha\beta}{R'}\right)^s; \tag{26}$$

in each case, in virtue of the assumption (ii) of Section 1.

The essential content of equations (24) to (26) is that, once the other variables appearing in them are assigned, $E(x_{ij}x_{kl} \mid s)$ is a function only of the differences $i - k$ and $j - l$. It is this fact which allows us to evaluate the limits of the quantities in (21) and (22) in a simple manner, in effect by holding one of the two freely variable points (ij), (kl) in a fixed position, say at the origin. Thus, let

$$E(\theta_n \mid s) = d_n^2 \delta_n^2 \sum_{i=-v_1}^{0} \sum_{j=-v_2}^{0} E\left(x_{ij} \sum_{k=i}^{n+i} \sum_{l=j}^{n+j} x_{kl} \mid s\right). \tag{27}$$

Owing to the remark just made, the expectation

$$E\left(x_{ij} \sum_{k=i}^{n+i} \sum_{l=j}^{n+j} x_{kl} \mid s\right) = E\left(x_{00} \sum_{k=0}^{n} \sum_{l=0}^{n} x_{kl} \mid s\right) \tag{28}$$

and it follows that

$$E(\theta_n \mid s) = (v_1 + 1)(v_2 + 1)\, d_n^2 \delta_n^2 E\left(x_{00} \sum_{k=0}^{n} \sum_{l=0}^{n} x_{kl} \mid s\right) \tag{29}$$
$$= [(v_1 + 1)(v_2 + 1)\, d_n \delta_n] \left[d_n \delta_n \sum_{k=0}^{n} \sum_{l=0}^{n} E(x_{00} x_{kl} \mid s) \right].$$

Of the two factors in the square brackets in (29), the first tends to $\Delta a \Delta b$ as n tends to infinity, and the second tends to the integral

$$\int_0^a \int_0^b f^s(\xi, \eta)\, d\xi d\eta \tag{30}$$

where

$$f(\xi, \eta) \equiv 1 - \frac{2\alpha\beta - (\alpha - \xi)(\beta - \eta)}{R'} \tag{31}$$

if both $0 \leq \xi \leq \alpha$ and $0 \leq \eta \leq \beta$, and

$$f(\xi,\eta) \equiv \left(1 - \frac{2\alpha\beta}{R'}\right) \tag{32}$$

otherwise. Thus the computation of the limit of $E(\theta_n \mid s)$ is straightforward. It remains to show that it differs from that of $E(Y_n W_n \mid s)$ in equation (21) by an infinitesimal which is of an order higher than the product $\Delta a \Delta b$.

Since the variables x_{ik} are capable only of the two values unity and zero the absolute value of the difference between the brackets in (21) and (27), that is, between

$$x_{ij} \sum_{k=0}^{n} \sum_{l=0}^{n} x_{kl} \text{ and } x_{ij} \sum_{k=i}^{n+i} \sum_{l=j}^{n+j} x_{kl}, \tag{33}$$

cannot be greater than $-n(i + j) \leq n(v_1 + v_2)$. It follows that

$$| E(Y_n W_n \mid s) - E(\theta_n \mid s) | \leq [d_n \delta_n (v_1 + 1)(v_2 + 1)][n \delta_n v_1 d_n + n d_n v_2 \delta_n]. \tag{34}$$

As n tends to infinity, the right hand side of (34) tends to the product

$$\Delta a \Delta b [b \Delta a + a \Delta b]; \tag{35}$$

whence

$$\lim_{\Delta a, \Delta b \to 0} \frac{1}{\Delta a \Delta b} \{\lim_{n \to \infty} E(\theta_n \mid s)\} = \lim_{\Delta a, \Delta b \to 0} \frac{1}{\Delta a} \frac{1}{\Delta b} E(YW \mid s) \tag{36}$$
$$= \int_0^a \int_0^b f^s(\xi, \eta)\, d\xi d\eta.$$

A very similar procedure will serve to evaluate the limit of $E(UV \mid s)/\Delta a \Delta b$. Here, we replace the two freely variable points (ij), (kl) by two semi-fixed points, one being restricted to the axis $O\xi$ and the other to the axis $O\eta$. More precisely. instead of considering $E(U_n V_n \mid s)$ in equation (22) we consider, say,

$$E(\phi_n \mid s) = d_n^2 \delta_n^2 \sum_{i=-v_1}^{0} \sum_{j=0}^{n} E\left(x_{ij} \sum_{k=i}^{n+i} \sum_{l=-v_2}^{0} x_{kl}\right) \tag{37}$$

and it is easy to see that

$$\lim_{n\to\infty} \mid E(U_n V_n \mid s) - E(\Phi_n \mid s) \mid \leq b(\Delta a)^2 (\Delta b), \tag{38}$$

so that the quantity (37) may be used in equations (13) and (20) in place of the quantity (22). However, since $E(x_{ij} x_{kl} \mid s)$ depends only on the differences $i - k$ and $j - l$,

$$E(x_{ij} \sum_{k=i}^{n+i} \sum_{l=-v_2}^{0} x_{kl}) = E\left(x_{0j} \sum_{k=0}^{n} \sum_{l=-v_2}^{0} x_{kl}\right) \tag{39}$$

and therefore

$$E(\phi_n \mid s) = \{d_n(v_1 + 1)\} \left\{d_n \delta_n^2 \sum_{j=0}^{n} E\left(x_{0j} \sum_{k=0}^{n} \sum_{l=-v_2}^{0} x_{kl} \mid s\right)\right\} \tag{40}$$

Further, and in the same way, we may replace the sum in (40), namely

$$\sum_{j=0}^{n} E\left(x_{0j} \sum_{k=0}^{n} \sum_{l=-v_2}^{0} x_{kl} \mid s\right) = \sum_{k=0}^{n} \sum_{l=-v_2}^{0} E\left(x_{kl} \sum_{j=0}^{n} x_{0j} \mid s\right) \tag{41}$$

by the simpler sum

$$\begin{aligned} \sum_{k=0}^{n} \sum_{l=-v_2}^{0} E\left(x_{kl} \sum_{j=l}^{n+l} x_{0j} \mid s\right) &= (v_2 + 1) \sum_{k=0}^{n} E\left(x_{k0} \sum_{j=0}^{n} x_{0j} \mid s\right) \\ &= (v_2 + 1 \sum_{k=0}^{n} \sum_{j=0}^{n} E(x_{k0} x_{0j} \mid s). \end{aligned} \tag{42}$$

It follows that we may replace the limit of $E(U_n V_n \mid s)$ as expressed in (22) by

$$\lim_{n\to\infty} \{d_n (v_1 + 1) \delta_n (v_2 + 1)\} \left\{d_n \delta_n \sum_{k=0}^{n} \sum_{j=0}^{n} E(x_{k0} x_{0j} \mid s)\right\}, \tag{43}$$

and this is easily found to be equal to

$$\Delta a \Delta b \int_0^a \int_0^b f^s (\xi, \eta)\, d\xi d\eta, \tag{44}$$

where $f(\xi, \eta)$ is defined by the formulae (31) and (32).

Collecting this result with that expressed by (36), and substituting in equation (13), we therefore have finally

$$D(a,b \mid s) = 4 \int_0^a \int_0^b f^s (\xi, \eta)\, d\xi d\eta. \tag{45}$$

3. The forms of the derivative. Since the function $f(\xi,\eta)$ has two different forms (31) or (32) depending on the relationships between a, b, α and β, it will be necessary to distinguish four different forms of the derivative (45), and of its integral.

First, for values of a and b for which simultaneously

$$a \leq \alpha \quad \text{and} \quad b \leq \beta, \tag{46}$$

the integrand in (45) has the form (31) for the whole region of integration. Hence the value of $D(a, b \mid s)$ in the region (46) is given by, say

$$\begin{aligned} D_1 &= 4\int_0^a\int_0^b\left(1 - \frac{2\alpha\beta - (\alpha - \xi)(\beta - \eta)}{R'}\right)^s d\xi d\eta \\ &= 4\int_{\alpha-a}^{\alpha}\int_{\beta-b}^{\beta} g^s(t,\tau)\,dt d\tau, \end{aligned} \tag{47}$$

where

$$g(t,\tau) \equiv 1 - \frac{2\alpha\beta - t\tau}{R'}. \tag{48}$$

Next, when $a \geqq \alpha$ but $b \leq \beta$, the integrand in (45) has the form determined by (31) only when

$$0 \leq \xi \leq \alpha, \qquad 0 \leq \eta \leq b, \tag{49}$$

whereas when

$$\alpha \leq \xi \leq a, \qquad 0 \leq \eta \leq b, \tag{50}$$

the appropriate form is that determined by (32). Therefore here $D(a, b \mid s)$ has the form, say,

$$D_2 = 4b(a - \alpha)\left(1 - \frac{2\alpha\beta}{R'}\right)^s + 4\int_0^{\alpha}\int_{\beta-b}^{\beta} g^s(t,\tau)\,dt d\tau, \tag{51}$$

Similarly, for

$$a \leq \alpha \quad \text{but} \quad b \geqq \beta, \tag{52}$$

$D(a, b \mid s)$ is given by, say,

$$D_3 = 4a(b - \beta)\left(1 - \frac{2\alpha\beta}{R'}\right)^s + 4\int_{\alpha-a}^{\alpha}\int_0^{\beta} g^s(t,\tau)\,dt d\tau. \tag{53}$$

Finally, in the region in which simultaneously

$$a \geqq \alpha \quad \text{and} \quad b \geqq \beta, \tag{54}$$

$D(a, b \mid s)$ has the form, say,

$$D_4 = 4(ab - \alpha\beta)\left(1 - \frac{2\alpha\beta}{R'}\right)^s + 4\int_0^{\alpha}\int_0^{\beta} g^s(t,\tau)\,dt d\tau. \tag{55}$$

4. The second moment of Y. We have now to determine $m(a, b \mid s)$ for all non-negative values of a and b, from the equation

$$\frac{\partial^2 m(a, b \mid s)}{\partial a\, \partial b} = D(a, b \mid s). \tag{56}$$

The general solution of this equation is

$$m(a, b \mid s) = \int_0^a \int_0^b D(a, b \mid s)\, da db + A(a) + B(b), \tag{57}$$

where $A(a)$ and $B(b)$ are each functions of one variable. These functions are determined by the boundary conditions, namely

$$m(a, 0 \mid s) = m(0, b \mid s) = \frac{\partial m(a, 0 \mid s)}{\partial a} = \frac{\partial m(0, b \mid s)}{\partial b} = 0, \tag{58}$$

which are a consequence of the inequality $0 \leq Y \leq ab$. It is then easily found that the only solution $m(a, b \mid s)$ satisfying (57) and (58) has the following four different forms, depending on the values of a and b.

If $a \leq \alpha$ and $b \leq \beta$, then

$$m(a, b \mid s) = \int_0^a \int_0^b D_1(x, y)\, dxdy = m_1(a, b \mid s) \quad \text{(say)}. \tag{59}$$

If $a \geqq \alpha$ and $b \leq \beta$, then

$$\begin{aligned} m(a, b \mid s) &= m_1(\alpha, b \mid s) + \int_\alpha^a \int_0^b D_2(x, y)\, dxdy \\ &= m_2(a, b \mid s) \quad \text{(say)}. \end{aligned} \tag{60}$$

If $a \leq \alpha$ and $b \geqq \beta$, then

$$\begin{aligned} m(a, b \mid s) &= m_1(a, \beta \mid s) + \int_0^a \int_\beta^b D_3(x, y)\, dxdy \\ &= m_3(a, b \mid s) \quad \text{(say)}. \end{aligned} \tag{61}$$

Finally, if $a \geqq \alpha$ and $b \geqq \beta$, then

$$\begin{aligned} m(a, b \mid s) = m_1(\alpha, \beta \mid s) &+ \int_\alpha^a \int_0^\beta D_2(x, y)\, dxdy + \int_0^\alpha \int_\beta^b D_3(x, y)\, dxdy \\ &+ \int_\alpha^a \int_\beta^b D_4(x, y)\, dxdy = m_4(a, b \mid s) \quad \text{(say)}. \end{aligned} \tag{62}$$

The procedure used to evaluate the integrals (59) to (62) follows the same general pattern, and we shall confine ourselves to outlining it in one case, say (59). There

$$\begin{aligned} m_1(a, b \mid s) &= \int_0^a \int_0^b D_1(x, y)\, dxdy \\ &= 4 \int_0^a \int_0^b dxdy \int_{\alpha - x}^{\alpha} \int_{\beta - y}^{\beta} g^s(t, \tau)\, dt\, d\tau \\ &= 4 \int_0^a dx \int_{\alpha - x}^{\alpha} dt \left\{ \int_0^b dy \int_{\beta - y}^{\beta} g^s(t, \tau)\, d\tau \right\}. \end{aligned} \tag{63}$$

Integrating the double integral in the braces by parts for y we get, say,

$$I(t) \equiv \int_0^b dy \int_{\beta-y}^{\beta} g^s(t, \tau)\, d\tau = \left[y \int_{\beta-y}^{\beta} g^s(t, \tau)\, d\tau \right]_0^b - \int_0^b y g^s(t, \beta - y)\, dy, \tag{64}$$

whence, substituting $\beta - y = \tau$ in the last integral,

$$\begin{aligned} I(t) &= b \int_{\beta-b}^{\beta} g^s(t, \tau)\, d\tau - \int_{\beta-b}^{\beta} (\beta - \tau) g^s(t, \tau)\, d\tau \\ &= \int_{\beta-b}^{\beta} (\tau + b - \beta) g^s(t, \tau)\, d\tau. \end{aligned} \tag{65}$$

Proceeding now in the same manner with the other double integration in (63), we conclude that

$$\begin{aligned} m_1(a, b \mid s) &= 4 \int_0^a dx \int_{\alpha-x}^{\alpha} I(t)\, dt = 4 \int_{\alpha-a}^{\alpha} (t + a - \alpha) I(t)\, dt \\ &= 4 \int_{\alpha-a}^{\alpha} dt \int_{\beta-b}^{\beta} (t + a - \alpha)(\tau + b - \beta) g^s(t, \tau)\, d\tau, \end{aligned} \tag{66}$$

where, throughout, $g(t, \tau)$ is defined by (48).

Formulae for $m_2(a, b \mid s)$, $m_3(a, b \mid s)$ and $m_4(a, b \mid s)$ are obtained by a similar procedure. They may conveniently be summarized in the following single expression. Define a symbol $[x]$ for any real number x by the equations

$$\begin{aligned} [x] &= x \quad \text{if} \quad x \geqq 0 \\ [x] &= 0 \quad \text{if} \quad x \leqq 0. \end{aligned} \tag{67}$$

With this notation, whatever be the relation between a, b, α and β, we have

$$\begin{aligned} m(a, b \mid s) &= 4 \int_{[\alpha-a]}^{\alpha} \int_{[\beta-b]}^{\beta} (t + a - \alpha)(\tau + b - \beta) \left\{ 1 - \frac{2\alpha\beta - t\tau}{R'} \right\}^s dt d\tau \\ &\quad + \{ a^2[b - \beta]^2 + b^2[a - \alpha]^2 - [a - \alpha]^2[b - \beta]^2 \} \left(1 - \frac{2\alpha\beta}{R'} \right)^s. \end{aligned} \tag{68}$$

We now allow s to take all values $s = 0, 1, 2, \cdots$ with probabilities P_s given by the generating function (1). Then it follows, from the form of (68), that

$$\begin{aligned} m(a, b) &= 4 \int_{[\alpha-a]}^{\alpha} \int_{[\beta-b]}^{\beta} (t + a - \alpha)(\tau + b - \beta) \Psi \left(1 - \frac{2\alpha\beta - t\tau}{R'} \right) dt d\tau \\ &\quad + \{ a^2[b - \beta]^2 + b^2[a - \alpha]^2 - [a - \alpha]^2[b - \beta]^2 \} \Psi \left(1 - \frac{2\alpha\beta}{R'} \right). \end{aligned} \tag{69}$$

On subtracting from this the square of the first moment of Y, which by (5) and (8) is

$$ab\Psi\left(1 - \frac{\alpha\beta}{R'}\right),$$

we obtain the variance σ_Y^2 of Y. But the variance of Y is necessarily equal to the variance σ_X^2 of X.

5. Particular cases. (i) $\Psi_1(u) = u^s$. This is the case, considered originally, in which the number s of centers of the rectangles ρ falling within R' is fixed. The explicit evaluation of the variance σ_X^2 depends in this case on the evaluation of the integral

$$\int_{[\alpha-a]}^{\alpha}\int_{[\beta-b]}^{\beta}(t + a - \alpha)(\tau + b - \beta)\left\{\left(1 - \frac{2\alpha\beta}{R'}\right) + \frac{t\tau}{R'}\right\}^s dt d\tau. \tag{70}$$

The evaluation is easy if one expands the binomial under the sign of the integral and integrates term by term. Each such integral is a product of two simple integrals.

(ii) $\Psi_2(u) = e^{\lambda R'(u-1)}$, *Poisson Case.* This is the case where the probabilities P_s that there are exactly s centers of rectangles ρ within R' are given by the Poisson Law, $P_s = (\lambda R')^s e^{-\lambda R'}/s!$. Substituting the expression of the probability generating function into (69), we obtain for this case

$$\begin{aligned} m(a, b) = 4e^{-2\alpha\beta\lambda}\int_{[\alpha-a]}^{\alpha}\int_{[\beta-b]}^{\beta}(t + a - \alpha)(\tau + b - \beta)\sum_{s=0}^{\infty}\frac{(\lambda t\tau)^s}{s!}\, dt d\tau \\ + e^{-2\alpha\beta\lambda}\{a^2[b - \beta]^2 + b^2[a - \alpha]^2 - [a - \alpha]^2[b - \beta]^2\}. \end{aligned} \tag{71}$$

On performing the integration term by term, and contracting the first term of the resulting infinite series into the second line of equation (71), we readily obtain the result

$$\begin{aligned} m(a, b) = 4e^{-2\alpha\beta\lambda}\sum_{s=1}^{\infty}\frac{(\lambda\alpha\beta)^s}{s!}\,\frac{\alpha\beta}{(s + 1)^2(s + 2)^2} \\ \times\left\{(s + 2)a - \alpha + [\alpha - a]\left(1 - \frac{a}{\alpha}\right)^{s+1}\right\} \\ \times\left\{(s + 2)b - \beta + [\beta - b]\left(1 - \frac{b}{\beta}\right)^{s+1}\right\} + e^{-2\gamma\beta\lambda}a^2 b^2, \end{aligned} \tag{72}$$

where $[x]$ continues to have the meaning defined by (67). In virtue of equations (7) and (8), however, the last term of the expression (72) is precisely the square of the first moment of Y when s is Poisson distributed. Hence, for s Poisson distributed, we have the expression for the variance of Y and of X,

$$\begin{aligned} \sigma_Y^2 = \sigma_X^2 = 4e^{-2\alpha\beta\lambda}\sum_{s=1}^{\infty}\frac{(\lambda\alpha\beta)^s}{s!}\,\frac{\alpha\beta}{(s + 1)^2(s + 2)^2} \\ \times\left\{(s + 2)a - \alpha + [\alpha - a]\left(1 - \frac{a}{\alpha}\right)^{s+1}\right\} \\ \times\left\{(s + 2)b - \beta + [\beta - b]\left(1 - \frac{b}{\beta}\right)^{s+1}\right\}. \end{aligned} \tag{73}$$

(iii) $\Psi_3(u) = e^{m(e^{\lambda R'(u-1)}-1)}$, *Contagious case.* This is the case where the probabilities P_s that there are exactly s centers of rectangles ρ within R' are given by the contagious law of type A with two parameters[2]. The evaluation of the second moment of Y is made easy by noticing that the probability generating function appropriate to the contagious distribution may be expressed as a series in terms of the probability generating function of the Poisson Law

$$\begin{aligned} \Psi_3(u) &= e^{-m} \sum_{k \geq 0} \frac{m^k}{k!} \Psi_2^k(u) \\ &= e^{-m} \sum_{k \geq 0} \frac{m^k}{k!} e^{k\lambda R'(u-1)}. \end{aligned} \tag{74}$$

Thus the evaluation of the integral intervening in the formula for the second moment of Y is reduced in the present case to that of formula (71).

6. Remarks on other cases. (i) It may be of interest, in amplification of H. E. Robbins' results, to exhibit the analogues of formulas (68), (69) and (73) in the one-dimensional case. Foı this case, then, if the interval a is embedded in a larger interval a', we obtain by similar methods beginning with the calculation of $\dfrac{\partial m(a \mid s)}{\partial a}$,

$$m(a \mid s) = 2 \int_{[\alpha-a]}^{\alpha} (t + a - \alpha)\left(1 - \frac{2\alpha - t}{a'}\right)^s dt + [a - \alpha]^2 \left(1 - \frac{2\alpha}{a'}\right)^s, \tag{75}$$

whence

$$m(a) = 2 \int_{[\alpha-a]}^{\alpha} (t + a - \alpha)\Psi\left(1 - \frac{2\alpha - t}{a'}\right) dt + [a - \alpha]^2 \Psi\left(1 - \frac{2\alpha}{a'}\right); \tag{76}$$

in particular, if s is Poisson distributed,

$$\begin{aligned} \sigma_X^2 = \sigma_Y^2 = 2e^{-2\alpha\lambda} \sum_{s=1}^{\infty} \frac{(\alpha\lambda)^s}{s!} \frac{\alpha}{(s+1)(s+2)} \\ \times \left\{(s+2)a - \alpha + [\alpha - a]\left(1 - \frac{a}{\alpha}\right)^{s+1}\right\}. \end{aligned} \tag{77}$$

The close parallel between these formulas and those for two dimensions make it natural to conjecture analogous formulas for n dimensions; but we have not attempted to establish such formulas.

(ii) For the evaluation of the higher moments of Y it may be useful to notice that precisely the same method as that described above leads to the conclusion that the derivative of the n-th non central moment of Y is

$$\frac{\partial^2 m_n(a, b)}{\partial a \partial b} = \lim_{\Delta a, \Delta b \to 0} \frac{1}{\Delta a \Delta b} \{nE(X^{n-1} W) + n(n-1)E(X^{n-2} UV)\}. \tag{78}$$

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[2] J. Neyman, "On a new class of contagious distributions", *Annals of Math. Stat.*, Vol. 10 (1939) pp. 35–57.