A NOTE ON CUMULATIVE SUMS

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Let $\{Z_i\}$ be a denumerable sequence of identical independent real-valued random variables. Two constants $a > 0 > b$ are chosen and the random variable n defined as the smallest integer for which one of the inequalities $\sum_1^n Z_i \geq a$ $\sum_1^n Z_i \leq b$ holds. For any events E_1 and E_2, $P\{E_1\}$ will denote the probability of the event E_1 and $P\{E_1 \mid E_2\}$ the conditional probability of the event E_1 given that E_2 has occurred.

It will be shown that there exists $t_0 > 0$ such that the moment generating function, Ee^{nt} exists for any complex number t whose real part is less than or equal to t_0, and as an immediate consequence that n has finite moments of all orders.

If d is any constant satisfying $b < d < a$, then, for fixed m,

$$P\left\{b < \sum_1^m Z_i + d < a\right\} \leq P\left\{\left|\sum_1^m Z_i\right| < c\right\} \tag{1}$$

where $c = |a| + |b|$. We exclude the case $P\{Z_i = 0\} = 1$. Then there exists $\epsilon > 0$ such that either

$$\delta_1 = P\{Z_i \geq \epsilon\} > 0 \quad \text{or} \quad \delta_2 = P\{Z_i \leq -\epsilon\} > 0.$$

Taking, for example, the former alternative with $m_1 = \left[\frac{c}{\epsilon}\right] + 1$,

$$P\left\{\left|\sum_1^{m_1} Z_i\right| \geq c\right\} \geq P\left\{Z_i \geq \epsilon \quad \text{for} \quad i = 1, \cdots, m_1\right\} = \delta_1^{m_1} > 0 \tag{2}$$

where $[w]$ denotes the largest integer less than or equal to w. For any poitive integer k,

$$\frac{P\{n > km_1\}}{P\{n > (k-1)m_1\}} = P\{n > km_1 \mid n > (k-1)m_1\}$$

$$\leq P\left\{b < \sum_1^{km_1} Z_i < a \,\middle|\, b < \sum_1^{s} Z_i < a \quad \text{for} \quad s = 1, \cdots, (k-1)m_1\right\}$$

since $n > km$ implies $b < \sum_1^{km_1} Z_i < a$.

But $\sum_1^{km_1} Z_i = \sum_1^{(k-1)m_1} Z_i + \sum_{(k-1)m_1+1}^{km_1} Z_i$ and the second sum on the right hand side is independent of all terms in the first sum.

Thus the distribution of $\sum_1^{km_1} Z_i$ given $\sum_1^{s} Z_i$ for $s = 1, \cdots, (k-1)m_1$ depends only on $\sum_1^{(k-1)m_1} Z_s$ so that

$$\text{(3)}\quad \begin{aligned} \frac{P\{n > km_1\}}{P\{n > (k-1)m_1\}} &\leq P\left\{b < \sum_{(k-1)m_1+1}^{km_1} Z_i + \sum_1^{(k-1)m_1} Z_i < a \,\middle|\, b < \sum_1^{(k-1)m_1} Z_i < a\right\} \\ &\leq P\left\{\left|\sum_{(k-1)m_1+1}^{km_1} Z_i\right| < c\right\} \leq 1 - \delta_1^{m_1} \quad \text{by (1) and (2).} \end{aligned}$$

Consequently, by induction on k,

$$\text{(4)}\qquad P\{n > m\} \leq P\left\{n > \left[\frac{m}{m_1}\right] m_1\right\} \leq (1 - \delta_1^{m_1})^{\left[\frac{m}{m_1}\right]}.$$

Let t_0 be any positive number less than $-\frac{1}{m_1}\log(1 - \delta_1^{m_1})$.

Then

$$\text{(5)}\quad \begin{aligned} Ee^{nt_0} &= \sum_{m=1}^{\infty} e^{mt_0} P\{n = m\} \\ &\leq \sum_{k=1}^{\infty} e^{km_1t_0} P\{(k-1)m_1 < n \leq km_1\} \\ &\leq \sum_{k=1}^{\infty} e^{km_1t_0} P\{n > (k-1)m_1\} \\ &\leq \sum_{k=1}^{\infty} e^{km_1t_0} (1 - \delta_1^{m_1})^{k-1} \\ &= \frac{1}{1 - \delta_1^{m_1}} \sum_{k=1}^{\infty} \{e^{m_1t_0}(1 - \delta_1^{m_1})\}^k. \end{aligned}$$

But this is a geometric series with decreasing terms, and is consequently convergent. Thus for any t whose real part $R(t) \leq t_0$, the moment generating function Ee^{nt} exists. Since, for all positive l, $m^l < e^{mt_0}$ for sufficiently large m, n has finite moments of all orders.