SOME IMPROVEMENTS IN SETTING LIMITS FOR THE EXPECTED NUMBER OF OBSERVATIONS REQUIRED BY A SEQUENTIAL PROBABILITY RATIO TEST

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Summary. Upper and lower limits for the expected number n of observations required by a sequential probability ratio test have been derived in a previous publication [1]. The limits given there, however, are far apart and of little practical value when the expected value of a single term z in the cumulative sum computed at each stage of the sequential test is near zero. In this paper upper and lower limits for the expected value of n are derived which will, in general, be close to each other when the expected value of z is in the neighborhood of zero. These limits are expressed in terms of limits for the expected values of certain functions of the cumulative sum Z_n at the termination of the sequential test.

In section 7 a general method is given for determining limits for the expected value of any function of Z_n.

1. Introduction. Let x be a random variable and let $f(x, \theta)$ be the elementary probability law of x involving an unknown parameter θ. Let H_0 denote the hypothesis that $\theta = \theta_0$, and H_1 the hypothesis that $\theta = \theta_1$, where θ_0 and θ_1 are given specified values. The sequential probability ratio test for testing H_0 against H_1, as defined in [1], is given as follows: Put

$$z_i = \log \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)} \tag{1.1}$$

where x_i denotes the i-th observation on x. Two constants, a and b are chosen where $a > 0$ and $b < 0$. At each stage of the experiment, at the m-th trial for each positive integral value m, the cumulative sum

$$Z_m = z_1 + \cdots + z_m \tag{1.2}$$

is computed. Experimentation is continued as long as $b < Z_m < a$. The first time that Z_m does not lie between b and a, experimentation is terminated. The hypothesis H_1 is accepted if $Z_m \geqq a$, and H_0 is accepted if $Z_m \leqq b$.

Let n denote the smallest value of m for which Z_m does not lie between b and a. Then n is the number of observations required by the sequential test. The expected value of n is a function of the true parameter value θ and is denoted by $E_\theta(n)$.

Upper and lower limits for $E_\theta(n)$ have been derived in section 4 of [1]. These limits, however, are of little practical value when the expected value of

$$z = \log \frac{f(x, \theta_1)}{f(x, \theta_0)} \tag{1.3}$$

is in the neighborhood of zero, for they converge to $+\infty$ and $-\infty$, respectively, as the expected value of z approaches zero. It can be shown that the expected value of z is negative when $\theta = \theta_0$, and positive when $\theta = \theta_1$.[1] Thus, if the expected value of z is a continuous function of θ, there will be a value θ' between θ_0 and θ_1 such that the expected value of z is zero when $\theta = \theta'$. Hence, the limits for $E_\theta(n)$, as given in [1], are of no practical value when θ is near θ'.

The purpose of this paper is to derive upper and lower limits for $E_\theta(n)$ which will be, in general, close to each other when θ is in the neighborhood of θ'. Thus, it will generally be possible to obtain close limits for $E_\theta(n)$ over the whole range of θ, if the limits given here are used for values in a certain small interval containing θ', and the limits given in [1] are used when θ is outside this interval.

2. Notation. We shall use the following notations throughout the paper. For any random variable u, the symbol $E_\theta(u)$ will denote the expected value of u when θ is the true value of the parameter. The conditional expected value of u, under the restriction that some relationship R is fulfilled will be denoted by $E_\theta(u \mid R)$. The symbol $P(R \mid \theta)$ will denote the probability that the relationship R holds when θ is true.

The cumulative distribution function of z will be denoted by $F(z, \theta)$ when θ is the true value of the parameter. The moment generating function of z, when θ is true, will be denoted by $\varphi(t, \theta)$, i.e.

$$\varphi(t, \theta) = \int_{-\infty}^{\infty} e^{tz}\, dF(z, \theta). \tag{2.1}$$

3. Assumptions concerning the family of distribution functions $F(z, \theta)$. In this section we shall formulate two assumptions concerning $F(z, \theta)$ which will then be used to prove various lemmas and theorems. Since we are interested in values of θ near θ', we shall restrict the domain of θ to a finite closed interval I containing θ' in its interior. It will be understood throughout the paper that any statements concerning θ refer to the domain I, even if this is not explicitly stated.

ASSUMPTION 1. *The moment generating function $\varphi(t, \theta)$ exists for any point t in the complex plane and any value θ, and is a continuous function of θ.*

ASSUMPTION 2. *There eists a positive δ such that $P(e^z > 1 + \delta \mid \theta)$ and $P(e^z < 1 - \delta \mid \theta)$ have positive lower bounds with respect to θ.*

4. *Proof that $\varphi(t, \theta)$ is continuous in t and θ jointly and that all moments of z are continuous functions of θ.*[2] In this section we shall prove the following theorem:

[1] This follows easily from Lemma 1 in [1], p. 156.

[2] The original proof of the author was somewhat lengthy. The present proof was suggested by T. E. Harris.

THEOREM 4.1. *It follows from Assumption 1 that $\varphi(t, \theta)$ is continuous in t and θ jointly and all moments of z are continuous functions of θ.*

PROOF: First we show that $\varphi(t, \theta)$ is a bounded function of t and θ in the domain $|t| \leqq t_0$, for any finite positive value t_0. Clearly,

$$0 \leqq |\varphi(t, \theta)| \leqq 2[\varphi(t_0, \theta) + \varphi(-t_0, \theta)] \tag{4.1}$$

for all values t for which $|t| \leqq t_0$. The boundedness of $\varphi(t_0, \theta)$ and $\varphi(-t_0, \theta)$ follows from Assumption 1. Hence $\varphi(t, \theta)$ is a bounded function of θ and t over any bounded t-domain.

Let $\{t_m, \theta_m\}$ ($m = 1, 2, \cdots$, ad inf.) be a sequence of pairs converging to the pair (t', θ'). We have

$$\varphi(t_m, \theta_m) - \varphi(t', \theta') = [\varphi(t_m, \theta_m) - \varphi(t', \theta_m)] + [\varphi(t', \theta_m) - \varphi(t', \theta')]. \tag{4.2}$$

The second expression in brackets converges to zero by continuity in θ. Thus the first part of Theorem 4.1 is proved if we show that

$$\lim_{m \to \infty} [\varphi(t_m, \theta_m) - \varphi(t', \theta_m)] = 0. \tag{4.3}$$

It follows from Assumption 1 that for any given θ, $\varphi(t, \theta)$ is an analytic function with no singularities in any finite t-domain. Hence we can expand $\varphi(t_m, \theta_m)$ in a Taylor series around $t = t'$, i.e.

$$\varphi(t_m, \theta_m) - \varphi(t', \theta_m) = \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{\partial^k \varphi(t, \theta_m)}{\partial t^k} \bigg|_{t=t'} \right) (t_m - t')^k. \tag{4.4}$$

Let r be a given positive value. Because of the boundedness of $\varphi(t, \theta)$ in any finite t-domain, there exists a constant M such that $|\varphi(t, \theta)| < M$ for all θ and for all t in the domain $|t - t'| \leqq r$. From the Cauchy integral formula for an analytic function it follows that

$$\frac{1}{k!} \left| \frac{\partial^k \varphi(t, \theta_m)}{\partial t^k} \bigg|_{t=t'} \right| \leqq \frac{M}{r^k}. \tag{4.5}$$

From (4.4) and (4.5) we obtain

$$|\varphi(t_m, \theta_m) - \varphi(t', \theta_m)| \leqq M \sum_{k=1}^{\infty} \frac{|t_m - t'|^k}{r^k}. \tag{4.6}$$

Equation (4.3) is an immediate consequence of (4.6). This proves the first half of Theorem 4.1.

Let C be a circle in the complex t-plane with finite radius and center at the origin. According to the Cauchy integral formula we have

$$\frac{1}{2\pi i} \int_C \frac{\varphi(t, \theta)}{t^{k+1}} \, dt = \frac{1}{k!} \frac{\partial^k \varphi(t, \theta)}{\partial t^k} \bigg|_{t=0} = \frac{1}{k!} E_\theta(z^k). \tag{4.7}$$

Since $\varphi(t, \theta)$ is continuous in t and θ jointly, the integral on the left hand side of (4.7) is a continuous function of θ. This proves the second half of Theorem 4.1.

5. Some lemmas. In this section we shall prove several lemmas which will then be used to derive the results contained in sections 6 and 8.

LEMMA 5.1. *It follows from assumptions* 1 *and* 2 *that for any given* θ *the equation in* t

$$\varphi(t, \theta) = 1 \tag{5.1}$$

has exactly two real roots, one of which is zero. The other real root is different from zero if $E_\theta(z) \neq 0$. *If* $E_\theta(z) = 0$, *both roots are equal to zero, i.e., zero is a double root of* (5.1).

This lemma is essentially the same as Lemma 2 in [2] and the proof is therefore omitted.[3]

Let $h(\theta)$ denote the non-zero root of (5.1), if $E_\theta(z) \neq 0$. If $E_\theta(z) = 0$, we put $h(\theta) = 0$.

In what follows the variable t will be restricted to real values, unless the contrary is explicitly stated.

LEMMA 5.2. *It follows from assumptions* 1 *and* 2 *that* $h(\theta)$ *is a continuous function of* θ.

PROOF: It follows from assumption 2 that

$$\lim_{t \to \pm\infty} \varphi(t, \theta) = +\infty \tag{5.2}$$

uniformly in θ. Hence, since by definition

$$\varphi[h(\theta), \theta] = 1$$

identically in θ, $h(\theta)$ must be a bounded function of θ.

Let $\{\theta_m\}$ be a sequence of parameter values which converges to θ^*. From Theorem 4.1 it follows that

$$\lim_{m \to \infty} [\varphi(t, \theta_m) - \varphi(t, \theta^*)] = 0 \tag{5.3}$$

uniformly in t over any finite interval. Since $h(\theta)$ is bounded, we obtain from (5.3)

$$\lim_{m \to \infty} \{\varphi[h(\theta_m), \theta_m] - \varphi[h(\theta_m), \theta^*]\} = 0. \tag{5.4}$$

Since $\varphi[h(\theta_m), \theta_m] = 1$, it follows from (5.4) that

$$\lim_{m \to \infty} \varphi[h(\theta_m), \theta^*] = 1.$$

It follows from assumption 1 that for any limit point h of the bounded sequence $\{h(\theta_m)\}$ $(m = 1, 2, \cdots$, ad inf.) we have

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[3] Condition IV of Lemma 2 in [2] is not postulated here, since the validity of this condition is implied by assumption 1. Condition IV could have been omitted also in [2], since it follows from condition III.

$$\varphi(h, \theta^*) = 1 \tag{5.5}$$

If $h(\theta^*) = 0$, then equation $\varphi(t, \theta^*) = 1$ has the only root $t = 0$. Consequently, all limit points of $\{h(\theta_m)\}$ must be equal to zero, that is

$$\lim_{m\to\infty} h(\theta_m) = 0 \quad \text{if} \quad h(\theta^*) = 0. \tag{5.6}$$

Now let us assume that $h(\theta^*) \neq 0$. Since the second derivative of $\varphi(t, \theta)$ with respect to t is positive, it can be seen that $\varphi(t, \theta) < 1$ for values t in the open interval $(0, h(\theta))$, and $\varphi(t, \theta) > 1$ for any t outside the closed interval $[0, h(\theta)]$. Hence, $\varphi(t, \theta) < 1$ implies that $|h(\theta)| > |t|$ and $h(\theta)$ and t have the same sign. Now let t_0, be a value in the open interval $(0, h(\theta^*))$. Then we have

$$\varphi(t_0, \theta^*) < 1 \tag{5.7}$$

It follows from assumption 1 that

$$\varphi(t_0, \theta_m) < 1 \tag{5.8}$$

for sufficiently large m. Hence $h(\theta_m)$ and t_0 have the same sign and

$$|h(\theta_m)| > |t_0| \tag{5.9}$$

Inequality (5.9) implies that zero cannot be a limit point of the sequence $\{h(\theta_m)\}$. Since $\varphi(t, \theta^*) = 1$ has only the roots $t = 0$ and $t = h(\theta^*)$, it follows from (5.5) that the sequence $\{h(\theta_m)\}$ cannot have a limit point different from $h(\theta^*)$. Thus,

$$\lim_{m\to\infty} h(\theta_m) = h(\theta^*) \tag{5.10}$$

and Lemma 5.2 is proved.

LEMMA 5.3. *It follows from assumption* 1 *that for any given* t, $E_\theta(e^{|zt|})$ *is a bounded function of* θ.

PROOF: We have

$$E_\theta(e^{|tz|}) \leqq E_\theta(e^{tz} + e^{-tz}) = \varphi(t, \theta) + \varphi(-t, \theta) \tag{5.11}$$

It follows from assumption 1 that $\varphi(t, \theta)$ and $\varphi(-t, \theta)$ are bounded functions of θ. Hence Lemma 5.3 is proved.

LEMMA 5.4. *Let* θ' *be a value of* θ *such that* $E_{\theta'}(z) = 0$, *but* $E_\theta(z) \neq 0$ *for all* $\theta \neq \theta'$ *in an open interval containing* θ'. *It follows from assumptions* 1 *and* 2 *that*

$$\lim_{\theta\to\theta'} \left(-\frac{2E_\theta(z)}{h(\theta)}\right) = E_{\theta'}(z^2). \tag{5.12}$$

PROOF: We have

$$e^{h(\theta)z} = 1 + h(\theta)z + \frac{[h(\theta)]^2}{2} z^2 + \frac{[h(\theta)]^3}{6} z^3 e^{uh(\theta)z} \tag{5.13}$$

where $0 \leqq u \leqq 1$. Hence

$$E_\theta(e^{h(\theta)z}) = 1 + h(\theta)E_\theta(z) + \frac{[h(\theta)]^2}{2} E_\theta(z^2) + \frac{[h(\theta)]^3}{6} E_\theta(z^3 e^{uh(\theta)z}). \tag{5.14}$$

Since $E_\theta(e^{h(\theta)z}) = 1$, we obtain from (5.14)

$$h(\theta)E_\theta(z) + \frac{[h(\theta)]^2}{2} E_\theta(z^2) + \frac{[h(\theta)]^3}{6} E_\theta(z^3 e^{uh(\theta)z}) = 0. \tag{5.15}$$

We shall consider only values θ for which $h(\theta) \neq 0$. For such values of θ, also $E_\theta(z) \neq 0$. Dividing (5.15) by $h(\theta)E_\theta(z)$, we obtain

$$1 + \frac{h(\theta)}{2E_\theta(z)} \left[E_\theta(z^2) + \frac{h(\theta)}{3} E_\theta(z^3 e^{uh(\theta)z}) \right] = 0. \tag{5.16}$$

Let t_0 be an upper bound of $|h(\theta)|$ with respect to θ. Then for a suitably chosen constant C we have

$$|z^3 e^{uh(\theta)z}| < Ce^{|t_0 z|} \tag{5.17}$$

From this and Lemma 5.3 it follows that $E_\theta(z^3 e^{uh(\theta)z})$ is a bounded function of θ.

Because of the continuity of $h(\theta)$ we have

$$\lim_{\theta \to \theta'} h(\theta) = 0. \tag{5.18}$$

Lemma 5.4 follows from (5.16), (5.18), the boundedness of $E_\theta(z^3 e^{uh(\theta)z})$ and the fact that $E_\theta(z^2)$ is a continuous function of θ and $E_{\theta'}(z^2) > 0$.

LEMMA 5.5. *From assumptions 1 and 2 it follows that for any given t, $E_\theta(e^{|tZ_n|})$ exists and is a bounded function of θ.*

PROOF: It is sufficient to show that $E_\theta(e^{tZ_n})$ is a bounded function of θ for any t, since

$$e^{|tZ_n|} \leqq e^{tZ_n} + e^{-tZ_n} \tag{5.19}$$

Clearly, e^{tZ_n} lies between $e^{bt+z_n t}$ and $e^{at+z_n t}$ Hence Lemma 5.5 is proved if we show that $E_\theta(e^{z_n t})$ is a bounded function of θ.

It follows from Assumption 2 that there exists a positive integer k and a positive constant g such that

$$P(|z_1 + \cdots + z_k| \geqq a - b \mid \theta) \geqq g \tag{5.20}$$

for all θ. For any positive integer m and for any real values $\lambda_1 < \lambda_2$ we have

$$\frac{P[(m-1)k < n \leqq mk \mid \theta]}{P[(m-1)k < n \mid \theta]} \geqq g \qquad (m = 1, 2, \cdots, \text{ad inf.}) \tag{5.21}$$

and

$$\frac{P[(m-1)k < n \leqq mk \;\&\; \lambda_1 \leqq z_n < \lambda_2 \mid \theta]}{P[(m-1)k < n \mid \theta]} \tag{5.22}$$

$$\leqq 1 - [1 - P(\lambda_1 \leqq z < \lambda_2 \mid \theta)]^k.$$

Hence

$$(5.23)\qquad \frac{P[(m-1)k < n \leqq mk \;\&\; \lambda_1 \leqq z_n < \lambda_2 \mid \theta]}{P[(m-1)k < n \leqq mk \mid \theta]} \leqq \frac{1-[1-P(\lambda_1 \leqq z < \lambda_2 \mid \theta]^k}{g}.$$

Multiplying (5.23) by $P[(m-1)k < n \leqq mk \mid \theta]$ and summing with respect to m we obtain

$$(5.24)\qquad P(\lambda_1 \leqq z_n < \lambda_2 \mid \theta) \leqq \frac{1-[1-P(\lambda_1 \leqq z < \lambda_2 \mid \theta)]^k}{g}.$$

From (5.24) it follows readily that

$$(5.25)\qquad \frac{P(\lambda_1 \leqq z_n < \lambda_2 \mid \theta)}{P(\lambda_1 \leqq z < \lambda_2 \mid \theta)}$$

is a bounded function of λ_1, λ_2 and θ. Let A be an upper bound of the ratio (5.25). Then

$$(5.26)\qquad E_\theta(e^{tz_n}) \leqq AE_\theta(e^{tz}) = A\varphi(t, \theta).$$

Because of Assumption 1, $\varphi(t, \theta)$ is a bounded function of θ. Hence also $E_\theta(e^{tz_n})$ is bounded and Lemma 5.5 is proved.

6. The limiting value of $E_\theta(n)$ when θ approaches a value θ' for which $E_{\theta'}(z) = 0$. In this section we shall prove the following theorem:

THEOREM 6.1. *Let θ' be a value of θ such that $E_{\theta'}(z) = 0$, but $E_\theta(z) \neq 0$ for all $\theta \neq \theta'$ in an open interval containing θ'. If assumptions* 1 *and* 2 *hold, we have*

$$(6.1)\qquad \lim_{\theta \to \theta'} \left[E_\theta(n) - \frac{E_\theta(Zn^2)}{E_{\theta'}(z^2)} \right] = 0.$$

PROOF: Consider the Taylor expansion

$$(6.2)\qquad e^{h(\theta)z_n} = 1 + h(\theta)Z_n + \frac{[h(\theta)]^2}{2} Z_n^2 + \frac{[h(\theta)]^3}{6} Z_n^3 e^{\lambda h(\theta) z_n}$$

where $0 \leqq \lambda \leqq 1$. It was shown in [2] (p. 286) that

$$(6.3)\qquad E_\theta e^{h(\theta)\, z_n} = 1.$$

Hence, taking expected values on both sides of (6.2), we obtain

$$(6.4)\qquad h(\theta)E_\theta(Z_n) + \frac{[h(\theta)]^2}{2} E_\theta(Z_n^2) + \frac{[h(\theta)]^3}{6} E_\theta(Z_n^3 e^{\lambda h(\theta) z_n}) = 0.$$

We consider only values of θ for which $E_\theta(z) \neq 0$. For such values, also $h(\theta) \neq 0$. Thus, we can divide both sides of (6.4) by $h(\theta)E_\theta(z)$. We then obtain

$$(6.5) \qquad \frac{E_\theta(Z_n)}{E_\theta(z)} + \frac{h(\theta)}{2E_\theta(z)}\left[E_\theta Z_n^2 + \frac{h(\theta)}{3} E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})\right] = 0.$$

It was shown in [1] (p. 142) that

$$(6.6) \qquad E_\theta(n) = \frac{E_\theta(Z_n)}{E_\theta(z)}.$$

Hence

$$(6.7) \qquad E_\theta(n) + \frac{h(\theta)}{2E_\theta(z)}\left[E_\theta(Z_n^2) + \frac{h(\theta)}{3} E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})\right] = 0.$$

Let t_0 be an upper bound of $| h(\theta) |$. Then for a properly chosen constant C we have

$$(6.8) \qquad | Z_n^3 e^{\lambda h(\theta) Z_n} | \leqq C e^{| t_0 Z_n |}$$

From this and Lemma 5.5 it follows that $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$ is a bounded function of θ. Since $\lim_{\theta \to \theta'} h(\theta) = 0$ and $E_\theta(Z_n^2)$ has a positive lower bound, Theorem 6.1 follows from 6.7, Lemma 5.4 and Theorem 4.1.

If $\lim_{\theta \to \theta'} E_\theta Z_n^2 = E_{\theta'} Z_n^2$, Theorem 6.1 gives[4]

$$(6.9) \qquad E_{\theta'}(n) = \frac{E_{\theta'}(Z_n^2)}{E_{\theta'}(z^2)}.$$

Limits for $E_{\theta'}(n)$ can be obtained by computing limits for $E_\theta(Z_n^2)$. In the next section we shall give a general method for obtaining limits for $E_\theta[\psi(Z_n)]$, where $\psi(Z_n)$ is any function of Z_n.

7. Determination of lower and upper limits for the expected value of any function of Z_n. Let $\psi(Z_n)$ be a function of Z_n. Limits for $E_\theta[\psi(Z_n)]$ may be determined as follows: First we determine limits for $E_\theta[\psi(Z_n) \mid Z_n \geqq a]$. Let r be a positive variable. Clearly, for any given value r we have

$$(7.1) \qquad E_{\text{-}\theta}(\psi Z_n) \mid Z_{n-1} = a - r \text{ and } Z_n \geqq a] = E_\theta[\psi(a - r + z) \mid z \geqq r]$$

From (7.1) we obtain the limits

$$(7.2) \qquad \begin{aligned} \underset{0<r<a-b}{\text{g.l.b.}}\ E_\theta[\psi(a - r + z) \mid z \geqq r] &\leqq E_\theta[\psi(Z_n) \mid Z_n \geqq a] \\ &\leqq \underset{0<r<a-b}{\text{l.u.b.}}\ E_\theta[\psi(a - r + z) \mid z \geqq r]. \end{aligned}$$

Limits for $E_\theta[\psi(Z_n) \mid Z_n \leqq b]$ can be obtained in a similar way. Again, let r be a positive variable. For any value of r we have

$$(7.3) \qquad E_\theta[\psi(Z_n) \mid Z_n \leqq b \text{ and } Z_{n-1} = b + r] = E_\theta[\psi(b + r + z) \mid z \leqq -r$$

Hence we obtain the limits

[4] The validity of (6.9) was shown by the author [3] using an entirely different method.

$$
(7.4)\quad \begin{aligned} \underset{0<r<a-b}{\text{g.l.b.}}\ E_\theta[\psi(b+r+z)\mid z \leqq -r] &\leqq E_\theta[\psi(Z_n)\mid Z_n \leqq b] \\ &\leqq \underset{0<r<a-b}{\text{l.u.b.}}\ E_\theta[\psi(b+r+z)\mid z \leqq -r]. \end{aligned}
$$

Since

$$
(7.5)\quad E_\theta[\psi(Z_n)] = P(Z_n \geqq a)E_\theta[\psi(Z_n)\mid Z_n \geqq a] + P(Z_n \leqq b)E_\theta[\psi(Z_n)\mid Z_n \leqq b],
$$

a lower (upper) limit for $E_\theta[\psi(Z_n)]$ can be obtained, by replacing the conditional expected values on the right hand side of (7.5) by their lower (upper) limits given in (7.2) and (7.4).

8. Limits for $E_\theta(n)$ when $h(\theta)$ is near but unequal to zero. Let θ' be a value of θ for which $h(\theta') = 0$. In this section we shall derive limits for $E_\theta(n)$ which will generally be close to each other for values θ in a small neighborhood of θ'.

From equation (6.7) we obtain

$$
(8.1)\quad E_\theta(n) = -\frac{h(\theta)}{2E_\theta(z)}\left[E_\theta Z_n^2 + \frac{h(\theta)}{3} E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})\right]
$$

where $0 \leqq \lambda \leqq 1$. Thus, limits for $E_\theta(n)$ can be obtained by deriving limits for $E_\theta Z_n^2$ and $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$. Limits for $E_\theta Z_n^2$ can be obtained by using the method described in section 7.

If θ is near θ', any crude limits for $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$ will serve the purpose, since, as has been shown in section 6, $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$ is bounded and $\lim_{\theta \to \theta'} h(\theta) = 0$.

Limits for $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$ can be obtained as follows: For simplicity, let us assume that $h(\theta) > 0$. Then

$$
(8.2)\quad Z_n^3 \leqq Z_n^3 e^{\lambda h(\theta) Z_n} \leqq Z_n^3 e^{h(\theta) Z_n} \qquad (h(\theta) > 0)
$$

Thus, to determine limits for $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$, it is sufficient to determine a lower limit for $E_\theta(Z_n^3)$ and an upper limit for $E_\theta(Z_n^3 e^{h(\theta) Z_n})$. The latter limits may be derived by using the method given in section 7.

If $h(\theta) < 0$, we have

$$
(8.3)\quad Z_n^3 \geqq Z_n^3 e^{\lambda h(\theta) Z_n} \geqq Z_n^3 e^{h(\theta) Z_n}
$$

and a similar procedure will yield the desired limits for $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$.

It should be emphasized that the limits of $E_\theta(n)$, as given in this section, can be expected to be close only if $h(\theta)$ is near zero. For values of θ for which $h(\theta)$ is not near zero, the limits of $E_\theta(n)$ given in [1] can be used.

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