any calculations. However, we have to note an important inferiority of the equiprobable deviation of X compared to the mean and the standard deviations of X. If one or the other of the last two deviations is zero, X is a fixed number (except for the case of the probability zero). This property seems requested by the intuitive meaning which we attribute to the dispersion, and to every measure or any mark of it. Now, the equiprobable deviation lacks this property. If, for instance, X has only three values: 0, 2, 1, the first two with the probability 0.249, and the last with the probability 0.502, the equiprobable deviation of X will be zero, whereas X will be equal to its typical value 1 only with a probability of 0.502, and not with a probability equal to unity. The same holds for any distribution for which there is a point with probability exceeding $\frac{1}{2}$.

Remark. The definitions of the mean and of the equiprobable position become meaningless in the case that $M(X, a)$, or $M(X, a)^2$, is infinite. However, we succeeded in surmounting the difficulty, and to reach definitions which are valid even in this case. If X is a number, the new definitions become equivalent to the classical definitions of the mean and equiprobable value. The proofs are given in two recent articles [5], [6].

REFERENCES

[1] E. J. Gumbel, "Definition of the probable error," *Annals of Math. Stat.*, Vol. 13 (1942)' p. 110–111.

[2] E. J. Gumbel, "Probable deviation," *Stat. Jour.*, City College, (N. Y.), Vol. 6 (1943), pp. 25–26.

[3] M. Fréchet, "L'intégrale abstraite d'une fonction abstraite d'une variable abstraite et son application à la moyenne d'un élément aléatoire de nature quelconque," *Revue Scientifique*, Vol. 82 (1944), pp. 483–512.

[4] M. Fréchet, *Les Espaces Abstraites*, Gauthier-Villars, Paris, 1928, pp. 125–141.

[5] M. Fréchet, "Les éléments aléatoires de nature quelconque," *Ann. Inst. H. Poincaré*, 1947.

[6] M. Fréchet, "Nouvelles définitions de la valeur moyenne et des valeurs probables d'un nombre aléatoire," *Ann. Univ. de Lyon*, 1947.

THE GENERAL RELATION BETWEEN THE MEAN AND THE MODE FOR A DISCONTINUOUS VARIATE

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Dr. Gumbel has pointed out that one of the author's arguments employed in several particular cases (see [1]) can be employed in a general case which includes them and leads to the following result: If a statistical variate R has only positive entire values differing from zero, and if its mean value $\bar{R}$ is smaller than, or equal to, unity, the same holds for its equiprobable value $\bar{\bar{R}}$ and its mode $\tilde{R}$. There are two generalizations of this result which might be of interest:

1) On the one hand, the author has shown [2] that, if a variate R can only have values (entire or not) equal to, or larger than, zero, its equiprobable value

$\bar{\bar{R}}$ is, at most, equal to twice its mean value $\bar{R}$, and the inequality $\bar{\bar{R}}/\bar{R} \leqq 2$ cannot be improved which means that the upper boundary of the first member is exactly equal to (and not less than) two. The equality is reached when R has only two values of equal probability, one of them being zero.

2) On the other hand, if R is an integer positive variate equal to, or larger than zero, it can be proven that, if $\bar{R} \leqq \alpha$, we have

$$\tilde{R} \leqq \frac{\alpha(\alpha + 3)}{2}. \tag{1}$$

Here, $\bar{R}$ and $\tilde{R}$ stand for the mean and for the mode of R respectively, and α is a positive integer differing from zero. For example: if R is the number of repetitions of an event with probability p, we have, for n trials, $\bar{R} = np$, whence, if α is the first integer number equal to, or larger than, $\bar{R}$ we have the inequality (1) for the most probable number of repetitions. Naturally, this inequality only has an interest if the second member of (1) is smaller than n which means that

$$\alpha(\alpha + 3) < 2n.$$

This presupposes

$$2n > np(np + 3)$$

or

$$n < \frac{2 - 3p}{p^2}$$

and, since n must be positive,

$$p < \tfrac{2}{3}.$$

To prove the inequality (1), let us write ω_ν for the probability that $R = \nu$. We have

$$\sum_0^\infty \omega_\nu = 1; \qquad \sum_0^\infty \nu\omega_\nu = \bar{R} \leqq \alpha$$

whence

$$\sum_0^{\alpha-1} (\alpha - \nu)\omega_\nu \geqq \sum_{\alpha+1}^\infty (\nu - \alpha)\omega_\nu. \tag{2}$$

Let the mode be

$$\tilde{R} = \beta$$

then

$$\omega_\beta \geqq \omega_\nu\,; \; \nu = 0, 1, 2, \cdots$$

and the first member in (2) is bounded by

$$\frac{\alpha(\alpha + 1)}{2}\,\omega_\beta \geqq \sum_0^{\alpha-1} (\alpha - \nu)\omega_\nu. \tag{3}$$

Now, either $\alpha < \beta$ or $\beta \leqq \alpha$. In the first case the second member in (2) leads to

$$\sum_{\alpha+1}^{\infty} (\nu - \alpha)\omega_\nu \geqq (\beta - \alpha)\omega_\beta \tag{4}$$

since the second member in (4) is one of the terms occurring in the sum. The same inequality holds in the second case, $\beta \leqq \alpha$, hence it holds generally. It follows from (2), (3), and (4) that

$$\frac{\alpha(\alpha + 1)}{2} \omega_\beta \geqq (\beta - \alpha)\omega_\beta .$$

The probability ω_β is certainly different from zero, since $\sum_0^\infty \omega_\nu = 1$. Consequently

$$\beta - \alpha \leqq \frac{\alpha(\alpha + 1)}{2}$$

or

$$\beta \leqq \frac{\alpha(\alpha + 3)}{2}$$

as stated in (1).

The equality in (1) is possible only if, from (3),

$$\alpha(\omega_\beta - \omega_0) + (\alpha - 1)(\omega_\beta - \omega_1) + \cdots + (\omega_\beta - \omega_{\alpha-1}) = 0$$

and from (4)

$$\omega_{\alpha+1} + 2\omega_{\alpha+2} + \cdots + (\beta - \alpha)\omega_\beta + \cdots = (\beta - \alpha)\omega_\beta$$

whence

$$\omega_0 = \omega_1 = \cdots = \omega_\beta = \cdots = \omega_{\alpha-1} \tag{5}$$

and

$$\omega_{\alpha+1} = \omega_{\alpha+2} = \cdots = 0. \tag{5'}$$

The existence of the exceptional case proves that the inequality (1) cannot be improved by replacing the second member by a smaller function of α. In the exceptional case, the only possible values of R are

$$R = 0, 1, 2, \cdots, \alpha - 1, \alpha, \beta,$$

and all values, except perhaps α, are equiprobable. The probability ω_α may be, but need not be, equal to ω_β.

Moreover

$$\beta = \frac{\alpha(\alpha + 3)}{2} \geqq \alpha \tag{6}$$

and $\beta = \alpha$ is possible only if $\alpha = \beta = 0$ whence, from (5), $\omega_\nu = 0$ except for $\nu = 0$ which means that R only has one value equal to zero. Except for this trivial case, we have in the exceptional case $\beta > \alpha$, and there are $\alpha + 2$ possible values for R. Then we must have

$$\omega_\beta \geqq \omega_\alpha; \qquad \sum_0^\alpha \omega_\nu + \omega_\beta = 1$$

whence

$$(\alpha + 1)\omega_\beta + \omega_\alpha = 1$$

and, from (5),

$$\alpha \geqq \bar{R} = \omega_\beta \sum_1^{\alpha-1} \nu + \beta\omega_\beta + \alpha\omega_\alpha = \omega_\beta \left(\frac{\alpha(\alpha - 1)}{2} + \frac{\alpha(\alpha + 3)}{2} \right) + \alpha\omega_\alpha$$

$$= \alpha((\alpha + 1)\omega_\beta + \omega_\alpha)$$

whence

$$\bar{R} = \alpha. \tag{7}$$

From

$$1 = (\alpha + 1)\omega_\beta + \omega_\alpha \geqq (\alpha + 2)\omega_\alpha$$

follows

$$\omega_\alpha \leqq \frac{1}{\alpha + 2}; \qquad \omega_\beta = \frac{1 - \omega_\alpha}{\alpha + 1}. \tag{8}$$

These conditions (5), (5′), and (7) are necessary and sufficient for the existence of the exceptional case.

If the equality in (1) is excluded, the mode β and the smallest integer number α which is equal to, or larger than, the mean, are related by

$$\beta \leqq \frac{\alpha(\alpha + 3)}{2} - 1 = \frac{\alpha^2 + 3\alpha + 2}{2}. \tag{9}$$

As shown before, this general inequality, valid for any discontinuous variate, which can assume only non-negative integer values, cannot be improved without assuming specific properties of the distribution.

REFERENCES

[1] M. FRÉCHET, *Les probabilités associées à un système d'evénements compatibles et dependents*, Hermann et Cie., 1943, Part II, p. 5.

[2] M. FRÉCHET, "Comparaison de diverses mesures de la dispersion", *Rev. de l'Inst. International de Stat.*, Vol. 8 (1940), p. 5.