NOTES

This section is devoted to brief research and expository articles on methodology and other short items.

A REMARK ON CHARACTERISTIC FUNCTIONS

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1. Let $F(x)$, $-\infty < x < +\infty$, be a distribution function, and

$$\varphi(t) = \int_{-\infty}^{+\infty} e^{itx}\, dF(x)$$

its characteristic function. It is well known that the existence of $\varphi'(0)$ does not imply the existence of the absolute moment

$$\int_{-\infty}^{+\infty} |x|\, dF(x). \tag{1}$$

A simple example is provided by the function

$$\varphi(t) = C \sum_{n=2}^{\infty} \frac{\cos nt}{n^2 \log n},$$

where C is a positive constant. Since the series on the right differentiated term by term converges uniformly (see [1]), $\varphi'(t)$ exists (and is continuous) for all values of t, and in particular at the point $t = 0$. Obviously $\varphi(t)$ is the characteristic function of the masses $C/2n^2 \log n$ concentrated at the points $\pm n$ for $n = 2, 3, \cdots$. The constant C is such that the sum of all the masses is 1. The divergence of the series $\Sigma 1/n \log n$ implies that in this particular case the moment (1) is infinite.

In a recent paper (see [2], esp. p. 120, footnote), Fortet raises the problem of whether the existence of $\varphi'(0)$ implies the existence of the first algebraic moment

$$\int_{-\infty}^{+\infty} x\, dF(x) = \lim_{X \to +\infty} \int_{-X}^{X} x\, dF(x). \tag{2}$$

The main purpose of this note is to show that this is so. We shall even prove a slightly more general result.

A function $\psi(t)$ defined in the neighborhood of a point t_0 is said to be *smooth* at this point if

$$\lim_{h \to +0} \frac{\psi(t_0 + h) + \psi(t_0 - h) - 2\psi(t_0)}{h} = 0.$$

Clearly, if ψ has a one-sided derivative at the point t_0, the derivative on the other side also exists and has the same value. Thus the graph of $\psi(t)$ has no angular point for $t = t_0$, and this explains the terminology. If $\psi'(t_0)$ exists and is finite, $\psi(t)$ is smooth for $t = t_0$. The converse is obviously false, since any function whose graph is symmetric with respect to $t = t_0$ is smooth at that point.

THEOREM 1. *If the characteristic function* $\varphi(t)$ *is smooth at the point* 0, *then a necessary and sufficient condition for the existence of* $\varphi'(0)$ *is the existence of the moment* (2). *The value of* (2) *is* $-i\varphi'(0)$.

In particular, the existence and finiteness of $\varphi'(0)$ implies the existence of (2). That the converse is false, is obvious. For if $a_0, a_1, a_2, \cdots$ are positive numbers and $a_0 + 2a_1 + 2a_2 + \cdots = 1$, then $\psi(t) = a_0 + 2\Sigma_1^\infty a_n \cos nt$ is the characteristic function of the distribution function $F(x)$ corresponding to masses concentrated at the integer points $\pm n$ and having the values a_n there. Owing to the symmetry of the masses, the number (2) exists, and is zero even if $\varphi(t)$ is non-differentiable for $t = 0$ (we may e.g. take for $\varphi(t)$ the Weierstrass non-differentiable function $C\,\Sigma_1^\infty a^n \cos b^n t$, where C is a suitable constant).

PROOF. We may write

$$\varphi(t) = \int_0^\infty \cos xt\, dG(x) + i\int_0^\infty \sin xt\, dG(x) = \psi_1(t) + i\psi_2(t)$$

where

$$G(x) = F(x) - F(-x),\ H(x) = F(x) + F(-x).$$

Thus

$$0 \le |\Delta H| \le \Delta G. \tag{3}$$

Since $\varphi(t)$ is smooth at the point 0, and since $\psi_1(t)$ is even, $\psi_2(t)$ odd,

$$0 = \lim_{h\to+0} \frac{\varphi(h) + \varphi(-h) - 2\varphi(0)}{h} = 2\lim \frac{\psi_1(h) - \psi_1(0)}{h}$$
$$= -2\lim_{h\to+0}\int_0^\infty \frac{1-\cos hx}{h}\, dG(x)$$

so that, replacing h by $2h$,

$$\int_0^\infty \frac{\sin^2 hx}{h}\, dG(x) \to 0 \qquad \text{as } h \to 0.$$

Since the integrand is positive we obtain successively

$$\int_0^{1/h} \frac{\sin^2 hx}{h}\, dG(x) = o(1),$$

$$\int_0^{1/h} \frac{\left(\frac{2}{\pi} hx\right)^2}{h}\, dG(x) = o(1),$$

$$\int_0^{1/h} x^2\, dG(x) = o(h^{-1}), \tag{4}$$

$$\int_{1/2h}^{1/h} x^2\, dG(x) = o(h^{-1}),$$

$$\int_{1/2h}^{1/h} dG(x) = o(h). \tag{5}$$

Since $\psi_1(t)$ is even, the smoothness of $\varphi(t)$, and so also of $\psi_1(t)$, at the point $t = 0$ implies that $\psi_1'(0)$ exists and is zero. If $h \to +0$,

$$\frac{\psi_2(h) - \psi_2(0)}{h} = \int_0^\infty \frac{\sin xh}{h}\, dH(x) = \int_0^{1/h} + \int_{1/h}^\infty = A_h + B_h,$$

$$|B_h| \leq h^{-1} \int_{1/h}^\infty |dH| \leq h^{-1}\left(\int_{1/h}^{2/h} dG + \int_{2/h}^{4/h} dG + \int_{4/h}^{8/h} dG + \cdots\right)$$
$$= h^{-1} o(h + h/2 + h/4 + \cdots) = o(1),$$

by (3) and (5). Also

$$A_h - \int_0^{1/h} x\, dH = \int_0^{1/h} \left(\frac{\sin hx}{hx} - 1\right) x\, dH = \int_0^{1/h} O(x^2 h^2) x\, dG$$
$$= \int_0^{1/h} O(x^2 h)\, dG = o(1),$$

by (3) and (4). Thus

$$\frac{\psi_2(h) - \psi_2(0)}{h} = o(1) + \int_0^{1/h} x\, dH = o(1) + \int_{-1/h}^{1/h} x\, dF,$$

and so

$$\frac{\varphi(h) - \varphi(0)}{h} = o(1) + i \int_{-1/h}^{1/h} x\, dF.$$

It follows that the existence of (2) is equivalent to the existence of the right-hand side derivative of $\varphi(t)$ at the point $t = 0$, or, on account of smoothness, to the existence of $\varphi'(0)$. Moreover, the value of (2) is $-i\varphi'(0)$. This completes the proof of Theorem 1.

2. Suppose that a function $\psi(t)$ defined near the point t_0 satisfies for $h \to 0$ a relation

$$\psi(t_0 + h) = \alpha_0 + \alpha_1 h/1! + \cdots + \alpha_{k-1} h^{k-1}/(k-1)! + [\alpha_k + \sigma(1)]h^k/k!,$$

where $\alpha_0, \alpha_1, \cdots, \alpha_k$ are constants. Then α_k is called the *kth generalized derivative* of ψ at the point t_0. It will be denoted by $\psi_{(k)}(t_0)$. The existence and finiteness of $\psi^{(k)}(t_0)$ implies the existence of $\psi_{(k)}(t_0)$ and both numbers are equal.

Another generalization of higher derivatives is based on the consideration of the symmetric differences

$$\Delta_h \psi(t_0) = \psi(t_0 + h) - \psi(t_0 - h),$$
$$\Delta_h^2 \psi(t_0) = \psi(t_0 + 2h) - 2\psi(t_0) + \psi(t_0 - 2h),$$
$$\Delta_h^3 \psi(t_0) = \psi(t_0 + 3h) - 3\psi(t_0 + h) + 3\psi(t_0 - h) - \psi(t_0 - 3h).$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

If $\Delta_h^k \psi(t_0)/(2h)^k$ tends to a limit as $h \to +0$, this limit is called the kth symmetric derivative of ψ at the point t_0. We shall denote it by $D_k\psi(t_0)$. Clearly, $D_k\psi(t_0)$ exists and equals $\psi_{(k)}(t_0)$, if the latter number exists.

It is a simple matter to prove (see [3]) that if k is a positive even integer, and if the characteristic function $\varphi(t)$ has at $t = 0$ a finite symmetric derivative $D_k\varphi(0)$, then the kth moment $\int_{-\infty}^{+\infty} x^k \, dF(x)$ exists, and its value is $(-1)^{k/2} D_k\varphi(0)$. Conversely, the existence of $\int_{-\infty}^{+\infty} x^k \, dF(x)$ obviously implies (for k even) the existence and continuity of $\varphi^{(k)}(t)$ for all t, and in particular at the point $t = 0$.

In order to obtain an extension of Theorem 1 to the case of derivatives of odd order, we have to generalize the notion of smoothness. We shall say that a function $\psi(t)$ satisfies for $t = t_0$ condition S_k, $(k = 1, 2, \cdots)$, if

$$\Delta_h^{k+1}\psi(t_0) = o(h^k) \qquad \text{as } h \to +0.$$

For $k = 1$, condition S_k is identical with smoothness at t_0. Clearly, if $\psi_{(k)}(t_0)$ exists, ψ satisfies condition S_k at t_0.

THEOREM 2. *Suppose that k is a positive odd integer, and let $\varphi(t)$ be the characteristic function of a distribution function $F(x)$. If φ satisfies condition S_k at the point* 0, *a necessary and sufficient condition for the existence of $D_k\varphi(0)$ is the existence of the symmetric moment*

$$\int_{-\infty}^{\infty} x^k \, dF(x) = \lim_{X \to +\infty} \int_{-X}^{X} x^k \, dF(x) \tag{6}$$

whose value is then equal to $i^{-k} D_k\varphi(0)$. In particular, the existence of $\varphi_{(k)}(0)$ implies that of (6).

The proof of Theorem 2 is analogous to that of Theorem 1. Let $G(x)$ and $H(x)$ have the same meaning as before. Since $k + 1$ is even, condition S_k at the point $t = 0$ gives

$$\Delta_h^{k+1}\varphi(0) = \int_{-\infty}^{+\infty} (e^{ixh} - e^{-ixh})^{k+1} \, dF(x) = 2^{k+1}(-1)^{(k+1)/2} \int_{-\infty}^{+\infty} (\sin xh)^{k+1} \, dF(x)$$

$$= 2^{k+1}(-1)^{(k+1)/2} \int_0^{\infty} (\sin xh)^{k+1} \, dG(x) = o(h^k),$$

so that

$$\int_0^{1/h} (\sin xh)^{k+1} \, dG(x) = o(h^k)$$

$$\int_0^{1/h} x^{k+1} \, dG(x) = o(h^{-1}) \tag{7}$$

$$\int_{1/2h}^{1/h} dG(x) = o(h^k). \tag{8}$$

On the other hand,

$$i^{-k}\frac{\Delta_h^k\varphi(0)}{(2h)^k} = \int_{-\infty}^{+\infty}\left(\frac{\sin xh}{xh}\right)^k x^k\,dF(x) = \int_0^{\infty}\left(\frac{\sin xh}{xh}\right)^k x^k\,dH(x)$$
$$= \int_0^{1/h} + \int_{1/h}^{\infty} = A_h + B_h,$$

say. Here

$$|B_h| \le h^{-k}\int_{1/h}^{\infty} dG(x) = h^{-k}\left[\int_{1/h}^{2/h} + \int_{2/h}^{4/h} + \cdots\right]$$
$$= h^{-k}\left[o(h^k) + o\left(\frac{h}{2}\right)^k + \cdots\right] = o(1),$$

by (8). Since

$$\left(\frac{\sin u}{u}\right)^k = \{1 + O(u^2)\}^k = \{1 + O(u)\}^k = 1 + O(u)$$

for small u, we immediately obtain

$$A_h - \int_0^{1/h} x^k\,dH(x) = \int_0^{1/h} O(hx^{k+1})\,dG(x) = o(1),$$

by (7). Collecting the results, we see that

$$i^{-k}\frac{\Delta_h^k\varphi(0)}{(2h)^k} - \int_0^{1/h} x^k\,dH(x) = i^{-k}\frac{\Delta_h^k\varphi(0)}{(2h)^k} - \int_{-1/h}^{1/h} x^k\,dF(x) = o(1),$$

which completes the proof of Theorem 2.

One more remark. By Theorem 2, the existence of the first moment is equivalent to the existence of the first symmetric derivative

$$D_{(1)}\varphi(0) = \lim_{h\to 0}[\varphi(h) - \varphi(-h)]/2h.$$

In Theorem 1 we have a corresponding result for ordinary first derivative

$$\varphi'(0) = \lim_{h\to 0}[\varphi(h) - \varphi(0)]/h.$$

There is no discrepancy here since at every point where φ is smooth the two notions of derivative are equivalent.

REFERENCES

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[3] Harald Cramér, *Mathematical Methods of Statistics*, Princeton Univ. Press, 1946, p. 90.