If we let $f_n(x) = h_n(x) + p_0(x)$, and $f(x) = p_0(x)$, then $\lim f_n(x) = f(x)$, but $c_n = \frac{1}{2}$ and $c = 1$, hence $\lim c_n \neq c$. Employing the assumption that $p_n(x)$ and $p(x)$ are densities we see

$$1/c_n = \int_R f_n(x)\, dx, \qquad 1/c = \int_R f(x)\, dx,$$

and hence $\lim c_n = c$ if and only if

$$\lim \int_R f_n(x)\, dx = \int_R \lim f_n(x)\, dx. \tag{13}$$

It follows that in such cases if we wish to establish a limiting distribution in the sense (1), we may either prove $\lim c_n = c$, or we may justify (13), say by producing a suitable dominating function, but we need not do both. No doubt the first alternative would be preferable at all but the most advanced levels of teaching or exposition.

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AN EXPLICIT REPRESENTATION OF A STATIONARY GAUSSIAN PROCESS

By M. Kac[1] and A. J. F. Siegert

Cornell University and *Syracuse University*

1. In a paper which will soon appear in the *Journal of Applied Physics* [1] the authors have introduced methods of calculating certain probability distributions which are of importance in the theory of random noise in radio receivers.

The complexity of the physical problem and occasional uses of heuristic reasonings may have obscured some of the mathematical points. For this reason the authors felt that it may be worth while to illustrate one of the basic ideas on a simple but important example.

2. A stationary Gaussian process is a one parameter family $x(t)$ of random variables such that:

(a). $x(t)$ is normally distributed; the mean and the variance being independent of t

(b). the joint probability distribution of $x(t_1), x(t_2), \cdots, x(t_r)$ is multivariate Gaussian whose parameters depend only on the differences $t_j - t_k$.

[1] John Simon Guggenheim Memorial Fellow.

We assume, for the sake of simplicity, that the process is normalized, i.e.,

$$E\{x(t)\} = 0, \qquad E\{x^2(t)\} = 1$$

and we define the correlation function $\rho(\tau)$ by the usual formula

$$\rho(\tau) = E\{x(t)x(t + \tau)\}.$$

It is then well known[2] that a distribution function $\sigma(u)$ exists such that for all τ

$$\rho(\tau) = \int_{-\infty}^{\infty} \cos u\tau \, d\sigma(u). \tag{1}$$

3. Let $0 \leq s, t \leq T$ and consider the symmetric kernel

$$K(s, t) = \rho(s - t).$$

The fact that $\sigma(u)$ is non-decreasing implies that the kernel $\rho(s - t)$ is quasi-definite, i.e., for every L^2 function $g(t)$ on $(0, T)$ one has

$$\int_0^T \int_0^T g(s)\rho(s - t)g(t) \, ds \, dt \geq 0.$$

Thus the eigenvalues of the integral equation

$$\int_0^T \rho(s - t)f(t) \, dt = \lambda f(s) \tag{2}$$

are non-negative. Moreover, denoting by λ_j the eigenvalues and by $f_j(t)$ the corresponding normalized eigenfunctions of (2) we have by the classical theorem of Mercer (see [4], in particular part 6 of Ch. I) that

$$\rho(s - t) = \sum_j \lambda_j f_j(s) f_j(t), \tag{3}$$

where the series on the right is absolutely and uniformly convergent. It should be noted that in virtue of (1) $\rho(\tau)$ is a continuous function.

4. Let now $G_1, G_2, G_3, \cdots$ be independent, normally distributed random variables each having mean 0 and variance 1.

Consider the series

$$\sum_j \sqrt{\lambda_j} G_j f_j(t). \tag{4}$$

Since for each t we have

$$\sum_j (\sqrt{\lambda_j} f_j(t))^2 = \sum_j \lambda_j f_j^2(t) = \rho(0) = 1,$$

we infer that for each t the series (4) converges in the mean to a random variable $x(t)$. Moreover, by a theorem of Kolmogoroff [5], the series (4) converges, for each t, to $x(t)$ with probability 1.

[2] See [2]. The theorem in question (in a somewhat different form) seems to have been first established by N. Wiener in [3].

Thus we may write

$$x(t) = \sum_j \sqrt{\lambda_j} G_j f_j(t). \tag{5}$$

It is now easy to show that $x(t)$ thus defined is a stationary Gaussian process in $(0, T)$ with the correlation function $\rho(\tau)$.

In fact,

$$E\{x(s)x(t)\} = \sum_j \lambda_j f_j(s) f_j(t) = \rho(s - t), 0 \leq s, t \leq T,$$

and conditions (a) and (b) of section 2 follow from the well known properties of linear combinations of independent Gaussian random variables. Of course, we are dealing here with infinite linear combinations but the mean convergence noted above, is sufficient to justify the extension to our case.

5. It is more illuminating to think of the random variables G_j as measurable functions $G_j(\omega)$ defined on an abstract set Ω in which a Lebesgue measure has been established (the measure of the whole space being 1).

The representation (5) can then be written in the equivalent form

$$x(t, \omega) = \sum_j \sqrt{\lambda_j} G_j(\omega) f_j(t). \tag{6}$$

The equality, as established in section 4, holds for every t in the sense of mean convergence. Moreover, by the theorem of Kolmogoroff cited above, and by Fubini's theorem the equality (6) holds for almost every pair (t, ω), $(0 \leq t \leq T)$, in the sense of ordinary convergence.

Furthermore by Mercer's theorem (remember that $\lambda_j \geq 0$)

$$\sum_j \lambda_j = \int_0^T \rho(s - s)\, ds = T$$

and hence

$$\sum_j \lambda_j E\{G_j^2\} = \sum_j \lambda_j \int_\Omega G_j^2(\omega) = \sum_j \lambda_j = T < \infty.$$

Thus

$$\sum_j \lambda_j G_j^2(\omega)$$

converges for almost every ω and therefore the series

$$\sum_j \sqrt{\lambda_j} G_j(\omega) f_j(t) \tag{7}$$

converges in the mean for almost every ω.

Combining this fact with the observation that (7) converges almost everywhere to $x(t, \omega)$ we see that, for almost every ω, the series (7) converges in the mean to $x(t, \omega)$ and that consequently

$$\int_0^T x^2(t, \omega)\, dt = \sum_j \lambda_j G_j^2(\omega) \tag{8}$$

for almost every ω.

It should be noted that (8) could not, in general, be derived by just appealing to Parseval's relation. The main reason is that Parseval's relation holds only for complete orthonormal systems whereas the orthonormal system $\{f_n(t)\}$ of eigenfunctions may fail to be complete. If the kernel $\rho(s - t)$ is positive-definite (in which case all the eigenvalues are positive instead of just non-negative) then it is known that the eigenfunctions form a complete set. This actually, happens to be the case in most physical applications.

6. An important application of (8) is the calculation of the characteristic function of the distribution function of the random variable

$$I = \int_0^T x^2(t, \omega)\, dt. \tag{9}$$

In fact,

$$E\{\exp (i\xi I)\} = \prod_j E\{\exp (i\xi\lambda_j G_j^2\} = \prod_j (1 - i\xi\lambda_j)^{-\frac{1}{2}}. \tag{10}$$

The probability density of I is the Fourier integral

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\xi I) \prod_j (1 - i\xi\lambda_j)^{-\frac{1}{2}}\, d\xi$$

which, unfortunately, in most cases cannot be calculated explicitely. If

$$\rho(\tau) = e^{-\beta|\tau|},$$

in which case the process is also Markoffian, the eigenvalues λ_j can be calculated explicitly[3] but in more complicated cases it is quite difficult to determine them.

7. If $\rho(\tau)$ is absolutely integrable and $\sigma(\mu)$ absolutely continuous then, setting

$$A(u) = \sigma'(u),$$

we have $A(u) \geq 0$ and

$$\rho(\tau) = \int_{-\infty}^{\infty} \cos u\tau A(u)\, du = \int_{-\infty}^{\infty} e^{iu\tau} B(u)\, du, \qquad B(u) = \frac{A(u) + A(-u)}{2}.$$

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[3] See [6], in particular section 4. We take this opportunity to correct two misprints in this note. In the last formula on p. 64 M should be replaced by N. Also the limits of integration in formula (6) should be 0, s and s, $p + q$ instead of 0, $p + q$ and 0, $p + q$.

The N.D.R.C. Report 14-305 to which a reference is made has been declassified in the meantime. It contains results which originated both [1] and the present note.

[4] These and related results were stated in the abstract [7] by M. Kac. The paper is now being prepared for publication.

It can then be shown[4] that

$$\lim_{T\to\infty} \frac{1}{T} \sum_j \lambda_j^2 = 2\pi \int_{-\infty}^{\infty} B^2(u)\, du = \int_{-\infty}^{\infty} \rho^2(\tau)\, d\tau$$

and

$$\lim_{T\to\infty} \frac{1}{T} \sum_j \lambda_j^3 = (2\pi)^2 \int_{-\infty}^{\infty} B^3(u)\, du.$$

It follows now by standard methods that the characteristic function of

$$\frac{1}{\sqrt{T}} \left\{ \int_0^T x^2(t)\, dt - T \right\} \tag{11}$$

approaches, as $T \to \infty$,

$$\exp\left(-\frac{\sigma^2}{2}\xi^2\right),$$

where

$$\sigma^2 = \int_{-\infty}^{\infty} \rho^2(\tau)\, d\tau.$$

Thus, as $T \to \infty$, the distribution of (11) becomes normal with mean 0 and variance σ^2.

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APPROXIMATE FORMULAS FOR THE RADII OF CIRCLES WHICH INCLUDE A SPECIFIED FRACTION OF A NORMAL BIVARIATE DISTRIBUTION

BY E. N. OBERG

University of Iowa

1. Introduction. Given the normal bivariate error distribution

$$\phi(x, y) = (1/2\pi\sigma_x \sigma_y)e^{-(x^2/2\sigma_x^2 + y^2/2\sigma_y^2)}. \tag{1}$$

The purpose of this paper is to present certain approximate formulas for the radii of circles whose centers are at the origin, which include a prescribed proportion, p, of errors. The formulas are, for given σ_x, σ_y, and p,