REFERENCES

[1] P. HARTMAN AND A. WINTNER, "On the spherical approach to the normal distribution law," *Amer. Jour. of Math.*, Vol. 62 (1940), pp. 759–779.
[2] E. K. HAVILAND, "On the distribution functions of the reciprocal of a function and of a function reduced mod 1," *Amer. Jour. of Math.*, Vol. 68 (1941), pp. 408–414.
[3] E. JAHNKE AND F. EMDE, *Tables of Functions with Formulae and Curves*, 3d ed. (1938), reprinted by G. E. Stechert and Co., New York.
[4] E. LANDAU, "Ueber eine trigonometrische Ungleichung," *Math. Zeit.*, Vol. 37 (1933), p. 36.
[5] H. LEBESGUE, *Leçons sur les séries trigonométriques*, Paris, 1905.
[6] P. LÉVY, *Calcul des probabilités*, Paris, 1925.
[7] H. WEBER, *Elliptische Funktionen und algebraische Zahlen*, Braunschweig, 1891.
[8] A. WINTNER, "On a class of Fourier transforms," *Amer. Jour. of Math.*, Vol. 58 (1936), pp. 45–90 and p. 425.
[9] F. ZERNIKE, "Wahrscheinlichkeitsrechnung und mathematische Statistik," *Handbuch der Physik*, Vol. 3 (1928), pp. 419–492.

A NOTE ON THE FUNDAMENTAL IDENTITY OF SEQUENTIAL ANALYSIS

BY G. E. ALBERT

U. S. Naval Ordnance Plant, Indianapolis

1. Introduction. Let $\{z_i\}$, $(i = 1, 2, 3, \cdots)$, be a sequence of real valued random variables identically distributed according to the cumulative distribution function $F(z)$. Define the sums $Z_N = z_1 + z_2 + \cdots + z_N$ for every positive integer N. Choose two positive constants a and b and define the random variable n as the smallest integer N for which one of the inequalities $Z_N \geqq a$ or $Z_N \leqq -b$ holds. The notations $P(u \mid F)$ and $E(u \mid F)$ will denote the probability of u and its expectation respectively assuming that F is the distribution of the z_i.

Wald [1] has established the results contained in the following lemmas.

LEMMA 1. *If the variance of $F(z)$ is positive, $P(n < \infty \mid F)$ equals one.*

LEMMA 2. *If there exists a positive number δ such that $P(e^z < 1 - \delta \mid F) > 0$ and $P(e^z > 1 + \delta \mid F) > 0$ and if the moment generating function $\varphi(t) = E(e^{zt} \mid F)$ exists for all real values of t, then $\varphi(t)$ has one and only one minimum at some finite value $t = t_0$. Moreover, $\varphi''(t) > 0$ for all real values of t.*

It is the purpose of this note to establish the following extension of the validity of certain results given by Wald [1], [2].

THEOREM.[1] *Under the conditions of Lemma 2 the identity*

$$E\{e^{Z_n t}[\varphi(t)]^{-n} \mid F\} = 1 \tag{1}$$

[1] Wald's results show (1) to be valid for all complex t in the domain over which $|\varphi(t)| \geqq 1$ and the validity of the differentiation clause for all real t in that domain. The importance of the present extension arises from the fact that, if $E(x \mid F) \neq 0$, then $0 < \varphi(t) < 1$ on a certain interval of the real axis.

is valid and may be differentiated with respect to t under the expectation sign any number of times for all real values of t.

PROOF. The notation t_0 will be used consistently to denote the t value at which $\varphi(t)$ has its minimum.

The proof of the theorem follows Wald's methods quite closely and certain of the results given in [1] and [2] will be used here without discussion.

Consider first the validity of (1). For an arbitrary positive integer N let P_N be the probability $P(n \leqq N \mid F)$ and let $E_N(u \mid F)$ and $E_N^*(u \mid F)$ denote the conditional expectations of u subject to the respective conditions $n \leqq N$ and $n > N$. Wald [1] has shown that for any finite real value of t

$$(2) \qquad P_N E_N\{e^{Z_n t}[\varphi(t)]^{-n} \mid F\} + (1 - P_N)[\varphi(t)]^{-N} E_N^*\{e^{Z_N t} \mid F\} = 1.$$

Since $\lim_{N=\infty} P_N E_N\{[\varphi(t)]^{-n}\exp(Z_n t)\}$ is the left member of the identity (1), it suffices to demonstrate that

$$(3) \qquad \lim_{N=\infty} (1 - P_N)[\varphi(t)]^{-N} E_N^*\{e^{Z_N t} \mid F\} = 0$$

for all real values of t.

Since $1 - P_N$ tends to zero with increasing N and the expected value E_N^* involved in (3) is bounded independently of N for any fixed t, the only source of difficulty in proving (3) lies in the fact that $\varphi(t)$ may be less than unity on an interval of the real axis. That difficulty is easily avoided by the following device. Define the function

$$(4) \qquad G(x) = [\varphi(t_0)]^{-1} \int_{-\infty}^{x} e^{z t_0}\, dF(z).$$

Obviously $G(x)$ is a distribution function whose moment generating function $\psi(t)$ exists for all real t. Its mean is zero and its variance is positive as will be seen from the equations $E(x \mid G) = \varphi'(t_0)/\varphi(t_0)$ and $E(x^2 \mid G) = \varphi''(t_0)/\varphi(t_0)$. It follows that $\psi(t)$ is never less than unity for real values of t.

Let Ω denote the space of all $z_1, \cdots, z_N$ and let $\Omega(n > N)$ be that subset of Ω on which $n > N$. One has

$$(1 - P_N)[\varphi(t)]^{-N} E_N^*\{e^{Z_N t} \mid F\}$$

$$= \frac{\int_{\Omega(n>N)} e^{Z_N t}\, dF(z_1) \cdots dF(z_N)}{\int_{\Omega} e^{Z_N t}\, dF(z_1) \cdots dF(z_N)} = \frac{\int_{\Omega(n>N)} e^{Z_N(t-t_0)}\, dG(z_1) \cdots dG(z_N)}{\int_{\Omega} e^{Z_N(t-t_0)}\, dG(z_1) \cdots dG(z_N)}$$

$$= (1 - Q_N)[\psi(s)]^{-N} E_N^*\{e^{Z_N s} \mid G\}$$

where $s = t - t_0$ and $Q_N = P(n \leqq N \mid G)$. By Lemma 1, $1 - Q_N$ tends to zero as N is increased. Thus, since $\psi(s) \geqq 1$ for all real t and the expected value $E_N^*\{e^{Z_N s} \mid G\}$ is bounded independently of N for a fixed t, the equation (3) holds for all real t.

The differentiability clause of the theorem requires the following modification of a very powerful theorem due to Charles Stein [3].

LEMMA 3. *Under the conditions of Lemma 2, if the minimum $\varphi(t_0)$ of $\varphi(t)$ is less than unity, there exists a positive number t_1 such that*

$$E\{\exp [nt_1 - n \log \varphi(t_0)] \mid F\} < \infty. \tag{5}$$

PROOF. If G is the distribution of the z_i, by Stein's theorem there exists a positive number t_1 such that $E(e^{nt_1} \mid G)$ is finite. Let $\Omega(n = N)$ denote the subset of Ω on which $n = N$. Then

$$\begin{aligned} P(n = N \mid G) &= \int_{\Omega(n=N)} dG(z_1) \cdots dG(z_N) \\ &= [\varphi(t_0)]^{-N} \int_{\Omega(n=N)} e^{t_0 z_N} \, dF(z_1) \cdots dF(z_N) \\ &\geqq P(n = N \mid F) \exp [\min\{at_0, -bt_0\} - N \log \varphi(t_0)]. \end{aligned}$$

It follows that

$$E\{\exp [nt_1 - n \log \varphi(t_0)] \mid F\} \leqq E\{e^{nt_1} \mid G\} \exp [-\min\{at_0, -bt_0\}]$$

and the lemma is proved.

To continue with the theorem, Wald's proof [2] suffices for the case in which $\varphi(t_0) \geqq 1$. Attention will be given only to the case $\varphi(t_0) < 1$. As pointed out in section 2 of [2], the differentiability clause of the theorem will be established if it can be shown that for any finite interval I of the real axis and any pair of integers r_1 and r_2 there exists a function $D_{r_1 r_2}(Z_n, n)$ such that for all t in I one has

$$D_{r_1 r_2}(Z_n, n) \geqq | n^{r_1} Z_n^{r_2} e^{Z_n t} [\varphi(t)]^{-n} | \tag{6}$$

and

$$E\{D_{r_1 r_2}(Z_n, n) \mid F\} < \infty. \tag{7}$$

On referring to Wald's proof and using the inequality $-\log \varphi(t) \leqq -\log \varphi(t_0)$ for all t in I, it is seen that there exists a constant C and a positive number t_2 such that the function

$$D_{r_1 r_2}(Z_n, n) \equiv C n^{r_1} [\varphi(t_0)]^{-n} (e^{Z_n t_2} + e^{-Z_n t_2})$$

satisfies (6) for all t in I. To establish (7) use the inequalities (2.4) and (2.6) in Wald [2] to obtain

$$\begin{aligned} &E\{D_{r_1 r_2}(Z_n, n) \mid F\} \\ &= C \sum_{N=1}^{\infty} P(n = N \mid F) N^{r_1} [\varphi(t_0)]^{-N} E_{n=N} \{e^{Z_n t_2} + e^{-Z_n t_2} \mid F\} \\ &\leqq C\{e^{at_2} l(t_2) + e^{-bt_2} l(-t_2)\} E\{ \exp [r_1 \log n - n \log \varphi(t_0)] \mid F\}. \end{aligned} \tag{8}$$

That (7) is indeed satisfied now follows from (5) and the finiteness of the function $l(t)$ since for a large enough integer M one has

$$\sum_{N=M}^{\infty} P(n = N \mid F) \exp [r_1 \log N - N \log \varphi(t_0)]$$

$$\leqq \sum_{N=M}^{\infty} P(n = N \mid F) \exp [Nt_1 - N \log \varphi(t_0)] < \infty.$$

Thus the expected value on the extreme right in (8) is finite. This completes the proof of the theorem.

REFERENCES

[1] A. WALD, "On cumulative sums of random variables," *Annals of Math. Stat.*, Vol. 15 (1944), pp. 283–285.
[2] A. WALD, "Differentiation under the expectation sign in the fundamental identity of sequential analysis," *Annals of Math. Stat.*, Vol. 17 (1946), pp. 493–496.
[3] CHARLES STEIN, "A note on cumulative sums," *Annals of Math. Stat.*, Vol. 17 (1946), pp. 498–499.

A SIGNIFICANCE TEST AND ESTIMATION IN THE CASE OF EXPONENTIAL REGRESSION

BY D. S. VILLARS[1]

United States Rubber Company, Passaic, N. J.

1. Introduction. The principal problem under consideration in this note may be described as follows. Consider a variate, z, whose distribution for a given value of a fixed variate, t, is:

$$f(z \mid t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(z-a+be^{-kt})^2/2\sigma^2} \tag{1.1}$$

where a, b, and k are real-valued parameters. The regression of z on t is exponential, for it follows from (1.1) that the expected value of z, given t, is:

$$E(z \mid t) = a - be^{-kt}. \tag{1.2}$$

On the basis of a random sample $0_N(z_1, t_1; z_2, t_2; \cdots; z_N, t_N)$ it is desired to test whether $k = 0$ or ∞. The problem of "fitting" a curve, $z = a - be^{-kt}$, to the sample (*i. e.* of estimating a, b, and k from the sample) will also be treated.

As an illustration of how the statistical problems described above arise in

[1] Present address, Jersey City Junior College, Jersey City, N. J.