We summarize (6) and (7) in the following table:

β/α	$\geqq 2(\sqrt{2} - 1)$	$= \eta < 2\sqrt{2} - 1$
$W(U)$	$\geqq 2(\sqrt{2} - 1)^2\alpha^2$	$\geqq \frac{2 - \eta}{2 + \eta}\eta\alpha^2$

Thus for Kolmogoroff's case ($\eta = 1$) we have $W(U) \geqq \frac{1}{3}\alpha^2$.

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APPROXIMATE WEIGHTS

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1. Summary. The greatest fractional increase in variance when a weighted mean is calculated with approximate weights is, quite closely, the square of the largest fractional error in an individual weight. The average increase will be about one-half this amount.

The use of weights accurate to two significant figures, or even to the nearest number of the form: 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 55, 60, 65, 70, 75, 80, 85, 90, or 95, that is to say, of the form $10(1)20(2)50(5)100 \times 10^r$ can thus reduce efficiency by at most $\frac{1}{4}$ percent, which is negligible in almost all applications.

2. Proof. Let the optimum weights be W_i, $i = 1, 2, \cdots, n$, with $W_i \geq 0$, where it is convenient to choose the normalization $\Sigma W_i = 1$. Let σ^2 be the variance of $\Sigma W_i x_i$, then the variance of each x_i must be σ^2/W_i, and since this is a weighted mean, the means of the x_i are the same.

Let the approximate weights be $W_i(1 + \lambda\theta_i)$, where $0 < \lambda < 1$ and $|\theta_i| \leq 1$, $i = 1, 2, \cdots, n$. Thus λ is the largest fractional error which may be made in the situation considered. We need the weak requirement $\lambda < 1$! The approximately weighted mean is

$$\frac{\sum W_i(1 + \lambda\theta_i)x_i}{\sum W_i(1 + \lambda\theta_i)} = \sum W_i \frac{1 + \lambda\theta_i}{1 + \lambda\bar{\theta}},$$

where $\bar{\theta} = \Sigma W_i\theta_i$. Its variance is

$$\sum W_i^2 \left(\frac{1 + \lambda\theta_i}{1 - \lambda\bar{\theta}}\right)^2 \frac{\sigma^2}{W_i}$$

$$= \sigma^2 \left\{1 + \frac{\lambda}{1 + \lambda\bar{\theta}} \sum W_i(\theta_i - \bar{\theta}) + \frac{\lambda^2}{(1 + \lambda\bar{\theta})^2} \sum W_i(\theta_i - \bar{\theta})^2\right\}$$

$$= \sigma^2 \left\{1 + \lambda^2 \frac{(\sum W_i\theta_i^2) - \bar{\theta}^2}{(1 + \lambda\bar{\theta})^2}\right\},$$

and, since $\Sigma W_i\theta_i^2 \leq 1$, this is bounded by

$$\sigma^2 \left\{1 + \lambda^2 \frac{1 - \bar{\theta}^2}{(1 + \lambda\bar{\theta})^2}\right\}.$$

Now the only maximum of this expression for $|\bar{\theta}| \leq 1$ occurs when $\bar{\theta} = -\lambda$, and the bound becomes

$$\sigma^2 \left(1 + \frac{\lambda^2}{1 - \lambda^2}\right) = \frac{\sigma^2}{1 - \lambda^2}.$$

This proves the first statement in the summary.

The greatest fractional change which occurs when a number is approximated by one of the form $10(1)20(2)50(5)100 \times 10^r$ is $5/105$, which occurs, for example, when $10.499999 \cdots$, is replaced by 10. The same estimate applies to an approximation to two significant figures. The variance is thus multiplied by a factor bounded by

$$1 + \frac{5^2}{105^2 - 5^2} \leq 1.0023,$$

which proves the second statement.

The use of a weight of the simpler form 10, 15, 20, 30, 40, 50, 70, times a power of ten is seen in the same way to lead to an increase in variance and a decrease in efficiency of at most $4\frac{1}{4}$ percent.

3. Comment. It is interesting to compare the 90 possible values for 2 significant figures, the 35 possible values for the numbers proposed above, which might be called *two curtailed significant figures*, and the 24 possible values for logarithmic spacing at interval $(1.05)^2$, all of which extend over one power of ten with the same maximum fractional error in rounding. The use of the curtailed scheme for critical tables of weights and weighting coefficients would save more than 60 percent of the entries needed for two complete significant figures.

This device applies equally well to other numbers of significant figures.