where the summations include the terms $E[x_i^2]$ and $E[x_{n-i+1}^2]$, respectively. But it is known [2] that the sample mean is the regular unbiased estimate of μ with minimum variance. Setting each w equal to $1/n$ and combining equivalent terms yields

$$\sum_{j=1}^{n} E[x_i x_j] + \tfrac{1}{2}n\lambda = 0, \qquad i = 1, 2, \cdots, n.$$

Summing from $i = 1$ to $i = n$, and employing the relationships discussed in the preceding paragraph, we obtain

$$n + \tfrac{1}{2}n^2\lambda = 0,$$

whence

$$\lambda = -2/n,$$

and

$$\sum_{j=1}^{n} E[x_i x_j] = 1, \qquad i = 1, 2, \cdots, n,$$

where the summation includes the term $E[x_i^2]$. This equation leads to the properties mentioned at the beginning of this paragraph. The same equation can be used to evaluate $E[x_1^2]$ and $E[x_2^2]$ in samples of size 3 or 4 from the distribution $N(0, 1)$, after the product-moments have been found.

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NOTE ON AN ASYMPTOTIC EXPANSION OF THE nTH DIFFERENCE OF ZERO

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This note gives an asymptotic expansion of the nth difference of zero. It is known that the Stirling number $S_{n,s}$ of the second kind is defined by

$$(1) \qquad n!S_{n,s} = \Delta^n 0^s = \sum_{x=0}^{n} (-1)^{n-x} \binom{n}{x} x^s.$$

We shall first show that the Stirling number $S_{n,n+k}$ can be expanded in the form

$$(2) \qquad S_{n,n+k} = \frac{n^{2k}}{2^k \cdot k!}\left[1 + \frac{f_1(k)}{n} + \frac{f_2(k)}{n^2} + \cdots + \frac{f_t(k)}{n^t} + O(n^{-t-1})\right], \quad (t < k)$$

where $f_1, f_2, \cdots, f_t$ are polynomials in k and whose coefficients can be found by means of the following lemmas.

The first lemma is due to B. F. Kimball, [1, (5.3)].

LEMMA 1. (Kimball) *Let q be a real number such that $n + q > 0$, and let $f(x) = x^{n+q}$. Then we can write $\Delta^n f(x)$ in the form*

$$(3)\qquad \Delta^n f(x) = f^{[n]}(x + \tfrac{1}{2}n)\left[1 + \sum_{m=1}^{\infty} \binom{q}{2m} \cdot \left(\frac{n}{2x + n}\right)^{2m} W(m, n)\right],$$

where the value of $W(m, n)$ is given by

$$(4)\qquad W(m, n) = B_{2m}^{-n}(-\tfrac{1}{2}n)/(\tfrac{1}{2}n)^{2m},$$

$B_\nu^{-n}(x)$ being a so-called Bernoulli polynomial of negative order which was first defined by Nörlund [2].

LEMMA 2. *Let the sum of all $\binom{n}{k}$ products of k different numbers taken from the set $(1, 2, \cdots, n)$ be denoted by $S_k(n)$. Then we can express it in the form*

$$(5)\qquad S_k(n) = \sum_{\rho=1}^{k} (-1)^{k+\rho} \lambda_\rho(k) \binom{n + \rho}{k + \rho},$$

where the coefficients $\lambda_1(k), \lambda_2(k), \cdots$ satisfy the recurrence relation

$$(6)\qquad (k + \rho)\lambda_{\rho-1}(k) + \rho \cdot \lambda_\rho(k) = \lambda_\rho(k + 1)$$

with $\lambda_0 \equiv 0$, $\lambda_1 \equiv 1$ and $\lambda_{k+1}(k) = 0$.

PROOF. Clearly, among all $\binom{n}{k+1}$ products of $(k + 1)$ numbers out of $(1, 2, \cdots, n)$, there are exactly $\binom{n-1}{k}$ products containing the greatest factor n. The sum of these products is therefore $n \cdot S_k(n - 1)$. Repeating this reasoning, we get

$$(7)\quad S_{k+1}(n) = n \cdot S_k(n - 1) + (n - 1) \cdot S_k(n - 2) + \cdots + (k + 1) \cdot S_k(k).$$

Evidently, (5) is true for $k = 1$. Suppose now that it is true for $k = k$. Then the right-hand side of (7) can be written as

$$\sum_{\mu=0}^{n-k-1} (n - \mu) \sum_{\rho=1}^{k} (-1)^{k+\rho} \lambda_\rho(k) \binom{n + \rho - \mu - 1}{k + \rho}$$
$$= \sum_{\rho=1}^{k} (-1)^{k+\rho} \lambda_\rho(k) \left[(k + \rho + 1)\binom{n + \rho + 1}{k + \rho + 2} - \rho\binom{n + \rho}{k + \rho + 1}\right]$$
$$= \sum_{\rho=1}^{k+1} (-1)^{k+\rho+1} [(k + \rho)\lambda_{\rho-1}(k) + \rho \cdot \lambda_\rho(k)] \binom{n + \rho}{k + \rho + 1}.$$

The lemma thus follows by induction on k.

The number $S_k(n)$ may be called a Stirling number of the first kind. By the lemma just proved, it is easy to find

$$S_2(n) = 3\binom{n+2}{4} - \binom{n+1}{3}$$

$$S_3(n) = 15\binom{n+3}{6} - 10\binom{n+2}{5} + \binom{n+1}{4}$$

$$(8)\quad S_4(n) = 105\binom{n+4}{8} - 105\binom{n+3}{7} + 25\binom{n+2}{6} - \binom{n+1}{5}$$

$$S_5(n) = 945\binom{n+5}{10} - 1260\binom{n+4}{9} + 490\binom{n+3}{8} - 56\binom{n+2}{7} + \binom{n+1}{6}.$$

We shall see that in order to compute the coefficients of $f_1(k), f_2(k), \cdots$, it is sufficient to compute the values of $W(m, n), \lambda_1(m), \lambda_2(m), \cdots, (m = 1, 2, \cdots, t)$.

Let $f(x) = x^{n+k}$. Then by lemma 1, we have

$$n!\,S_{n,n+k} = \left[\frac{d^n}{dx^n} f(x + \tfrac{1}{2}n)\right]_{x=0} \cdot \left[1 + \sum_{m=1}^{\infty} \binom{k}{2m} W(m, n)\right].$$

From the definition of $S_k(n)$ it is easily seen that

$$(n + k)(n + k - 1) \cdots (n + 1) = n^n + n^{n-1}S_1(k) + \cdots + nS_{k-1}(k) + S_k(k)$$

Hence we may write

$$\frac{1}{n!}\left[\frac{d^n}{dx^n} f(x + \tfrac{1}{2}n)\right]_{x=0} = \frac{n^{2k}}{2^k \cdot k!}\left(1 + \frac{S_1(k)}{n} + \frac{S_2(k)}{n^2} + \cdots + \frac{S_k(k)}{n^k}\right).$$

It is clear from Kimball's paper [1] that

$$\sum_{m=1}^{\infty} \binom{k}{2m} W(m, n) = \sum_{n=1}^{t} \binom{k}{2m} W(m, n) + O(n^{-t-1}).$$

Substituting, we obtain

$$\begin{aligned}(9)\quad S_{n,n+k} &= \frac{n^{2k}}{2^k \cdot k!}\left[1 + \sum_{m=1}^{t} \binom{k}{2m} W(m, n) + O(n^{-t-1})\right] \cdot \left[1 + \sum_{m=1}^{t} \frac{S_m(k)}{n^m} + O(n^{-t-1})\right] \\ &= \frac{n^{2k}}{2^k \cdot k!}\left[1 + \sum_{m=1}^{t} \binom{k}{2m} W(m, n) + O(n^{-t-1})\right] \cdot \left[1 + \sum_{m=1}^{t} \sum_{\rho=1}^{m} \binom{k+\rho}{m+\rho} \frac{\lambda_\rho(m)}{(-1)^\rho(-n)^m} + O(n^{-t-1})\right].\end{aligned}$$

The last expression shows that the asymptotic expansion (2) can be obtained by computing the numbers $\lambda_\rho(m)$, $W(m, n)$ with $1 \leq \rho \leq m \leq t$. For example, consider the case $t = 3$ and notice that [1, (2.13)]

$$B_2^{-n}\left(-\frac{n}{2}\right) = \frac{n}{12}, \qquad B_4^{-n}\left(-\frac{n}{2}\right) = \frac{n^2}{48} - \frac{n}{120}, \qquad B_6^{-n}\left(-\frac{n}{2}\right) = \frac{5n^3}{576} + O(n^2),$$

and that $\lambda_1 \equiv 1$, $\lambda_2(2) = 3$, $\lambda_2(3) = 10$, $\lambda_3(3) = 15$. Then by a straightforward calculation of the right-hand side of (9) and by comparison with (2), we find

$$f_1(k) = \frac{1}{3}(2k^2 + k)$$

$$(10) \qquad f_2(k) = \frac{1}{18}(4k^4 - k^2 - 3k)$$

$$f_3(k) = \frac{1}{810}(40k^6 - 60k^5 - 2k^4 - 63k^3 + 133k^2 - 48k).$$

Finally, combining (2) with the well-known Stirling's formula [3]

$$(11) \qquad n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left[1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + O(n^{-4})\right],$$

and noting (1), we obtain

$$(12.1) \qquad \Delta^n 0^{n+k} = \frac{\sqrt{2\pi n}}{k!}\left(\frac{n^2}{2}\right)^k\left(\frac{n}{e}\right)^n\left[1 + \frac{g_1(k)}{n} + \frac{g_2(k)}{n^2} + \frac{g_3(k)}{n^3} + O(n^{-4})\right]$$

where $g_1(k)$, $g_2(k)$, $g_3(k)$ are polynomials in k, viz.

$$g_1(k) = \frac{1}{12}(8k^2 + 4k + 1).$$

$$g_2(k) = \frac{1}{288}(64k^4 - 40k + 1).$$

$$(12.2) \qquad g_3(k) = \frac{1}{51840} 2560k^6 - 3840k^5 + 832k^4 - 4032k^3$$

$$+ 8392k^2 - 3732k - 139).$$

The asymptotic formula of $\Delta^n 0^{n+k}$ just derived is much better than a result previously obtained [4]. Moreover, it may be noted that the asymptotic expansion of $S_{n,n+k}$ may be made as sharp as desired, since in fact, for any prescribed $t > 1$, $\lambda_\rho(m)$ and $B_{2m}^{-n}(-\frac{1}{2}n)$, $(1 \leq m \leq t)$, may be easily computed by (6) and Kimball's [1, (2.12)] respectively.

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AN INEQUALITY FOR KURTOSIS

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1. Summary. It is well known that, if the fourth moment about the mean of a frequency distribution equals the square of the variance, then the frequencies are piled up at exactly two points, namely, the two points that are one standard deviation away from the mean. In this paper is developed a general inequality which describes the piling up of frequency around these two points for the case where the fourth moment exceeds the square of the variance. In a sense, it is shown how "U-shaped" a distribution must be according to its second and fourth moments.

2. An inequality. Let x be a random variable whose distribution has the following moments:

$$\mu = E(x); \sigma^2 = E(x - \mu)^2; (\alpha^2 + 1)\sigma^4 = E(x - \mu)^4. \tag{1}$$

α^2 is non-negative for any distribution, and its positive square root will be denoted by α. Let

$$t = (x - \mu)/\sigma. \tag{2}$$

It will be shown that, if λ is an arbitrary positive number, then

$$\text{Prob }\{1 - \lambda\alpha \leqq t^2 \leqq 1 + \lambda\alpha\} > 1 - \lambda^{-2}. \tag{3}$$

If λ is chosen so as to make the left member in the braces positive, then t^2 is bounded away from zero, and (3) becomes:

$$\text{Prob }\{\sqrt{1 - \lambda\alpha} \leqq |t| \leqq \sqrt{1 + \lambda\alpha}\} > 1 - \lambda^{-2}, \qquad (\lambda\alpha < 1). \tag{4}$$

For example, if $\alpha = .5$ and $\lambda = \sqrt{2}$, then (4) shows that the probability is greater than .50 that t is either between .54 and 1.30, or between -1.30 and $-.54$. If $\alpha = .2$ and $\lambda = 3$, then (4) shows that the probability is greater than .88 that t is either between .63 and 1.27, or between -1.27 and $-.63$. In general, the smaller α is, the greater the probability that t is in a small interval around $+1$ or -1. In particular, if $\alpha = 0$, then λ may be taken arbitrarily large, so that (4) shows that the probability is unity that $t = \pm 1$; this is the well known case referred to above.