and hence (21) becomes

$$\frac{f_n(x)}{h_n(x)} = 1 - b_1 x + \cdots, \tag{33}$$

which is significant for x near 0.

EXACT LOWER MOMENTS OF ORDER STATISTICS IN SMALL SAMPLES FROM A NORMAL DISTRIBUTON

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1. Summary. Exact means in samples of size ≤ 3, and exact second moments and product-moments in samples of size ≤ 4, are given in Table 1 in terms of π for order statistics selected from the normal distribution $N(0, 1)$. The derivation employs multiple integration and some general properties of the moments.

TABLE 1

Expected values of lower moments of order statistics, $x_i \geq x_{i+1}$, in samples of size n from the normal distribution $N(0, 1)$.

Moment	$n = 2$	$n = 3$	$n = 4$
$E[x_1]$	$1/\sqrt{\pi}$	$3/(2\sqrt{\pi})$	
$E[x_2]$		0	
$E[x_1^2]$	1	$1 + \sqrt{3}/(2\pi)$	$1 + \sqrt{3}/\pi$
$E[x_2^2]$		$1 - \sqrt{3}/\pi$	$1 - \sqrt{3}/\pi$
$E[x_1x_2]$	0	$\sqrt{3}/(2\pi)$	$\sqrt{3}/\pi$
$E[x_1x_3]$		$-\sqrt{3}/\pi$	$-(2\sqrt{3} - 3)/\pi$
$E[x_1x_4]$			$-3/\pi$
$E[x_2x_3]$			$(2\sqrt{3} - 3)/\pi$
σ_1^2	$1 - 1/\pi$	$1 - (9 - 2\sqrt{3})/(4\pi)$	
σ_2^2		$1 - \sqrt{3}/\pi$	
σ_{12}	$1/\pi$	$\sqrt{3}/(2\pi)$	
σ_{13}		$(9 - 4\sqrt{3})/(4\pi)$	

2. Introduction. The usefulness of the lower moments of order statistics for determining the moments of the range and for other purposes is well established. In small samples, however, computation of the moments by quadrature is laborious [1]. The values shown in Table 1 should therefore be helpful in problems requiring the use of these moments for samples of size ≤ 4, since the constant π has been evaluated to several hundred decimal places. Some of the methods used to obtain these results may also be useful in approximating or verifying the moments in larger samples.

3. Multiple integration. Let n random selections from the normal distribution $N(0, 1)$ be arranged in order of size so that

$$x_1 \geq x_2 \geq \cdots \geq x_n .$$

For samples of size 2, the means and product-moment are easily obtained from the general formula

$$E[x_i^k x_j^h] = n! \int_{-\infty}^{\infty} \int_{x_n}^{\infty} \int_{x_{n-1}}^{\infty} \cdots \int_{x_2}^{\infty} x_i^k x_j^h$$
$$f(x_1)f(x_2) \cdots f(x_n)\, dx_1\, dx_2 \cdots dx_n$$

where

$$f(x_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_i^2},$$

$E\,[x_i^k]$ being the special case where $h = 0$. Multiple integration can also be used to find any product-moment, $E\,[x_i x_{j\neq i}]$, for samples of size 3, the order of integration being changed at any stage where necessary.

For the means in samples of size 3 and the product-moments in samples of size 4, the integrals reduce to double integrals which can be evaluated from the equation

$$\int_{-\infty}^{\infty} \int_{t_2}^{\infty} e^{-(a^2t_1^2+b^2t_2^2)}\, dt_1\, dt_2 = \frac{\pi}{2ab}.$$

This equation follows from the fact that

$$\int_{-\infty}^{\infty} \int_{bt_2/a}^{\infty} \frac{ab}{\pi} e^{-(a^2t_1^2+b^2t_2^2)}\, dt_1\, dt_2$$

is equivalent to

$$\int_0^1 \int_{p_2}^1 dp_1\, dp_2 ,$$

while the function

$$\phi(t_2) = e^{-b^2t_2^2} \int_{t_2}^{bt_2/a} e^{-a^2t_1^2}\, dt_1$$

has the symmetrical property that $\phi\,(t_2) = -\phi(-t_2)$, whence

$$\int_{-\infty}^{\infty} \phi(t_2)\, dt_2 = 0.$$

4. Some properties of the moments. The most obvious property of the moments of order statistics in samples from the normal distribution $N(0, 1)$ is their symmetry; thus:

$$E\,[x_i] = -E\,[x_{n-i+1}],$$
$$E\,[x_i^2] = E\,[x_{n-i+1}^2],$$
$$E\,[x_i x_j] = E\,[x_{n-i+1} x_{n-j+1}].$$

When sample values from any parent distribution are numbered in order of random selection, x_i and $x_{j \neq i}$ are statistically independent of each other, and the expected value of a product $x_i^k x_j^h$ is the product of the expected values of x_i^k and x_j^h. Numbering in order of size has the effect of increasing some expected values and decreasing others, leaving the sum of expected values of a given type unchanged, so that in general,

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (E[x_i^k x_j^h]) = \binom{n}{2} E[x_0^k]E[x_0^h]$$

where x_0 is a random selection. In particular, this equation holds for the special cases ($k = 1, h = 1$), ($k = 1, h = 0$), and ($k = 2, h = 0$); so that in samples from the normal distribution $N(0, 1)$,

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (E[x_i x_j]) = \tfrac{1}{2}n(n - 1)(E[x_0])^2 = 0,$$

$$\sum_{i=1}^{n} (E[x_i]) = nE[x_0] = 0,$$

$$\sum_{i=1}^{n} (E[x_i^2]) = nE[x_0^2] = n.$$

The foregoing relationships lead immediately to the evaluation of $E\ [x_1x_2]$ and $E\ [x_1^2]$ in samples of size 2. (The generalization of these relationships was suggested by Professor John H. Smith, whose unpublished manuscript on sampling from a rectangular distribution has also been instructive.)

In samples from a normal distribution, the covariance of every order statistic with the sample mean is the same as the variance of the sample mean. This implies that the variance of the sample mean $\leq$ the variance of any order statistic, the ratio of one standard deviation to the other being equal to the coefficient of correlation between the sample mean and the order statistic. To derive these properties, consider the linear function

$$m = w_1X_1 + w_2X_2 + \cdots + w_nX_n$$

of the order statistics $X_1, X_2, \cdots, X_n$ in a sample selected from the normal distribution $N(\mu, \sigma)$ with unknown μ and σ. Let

$$x_i = (X_i - \mu)/\sigma, \qquad i = 1, 2, \cdots, n.$$

The conditions $w_1 + w_2 + \cdots + w_n = 1$ and $w_{n-i+1} = w_i$ are sufficient to make m an unbiased estimate of μ with variance $\sigma^2\ E\ [(w_1x_1 + w_2x_2 + \cdots + w_nx_n)^2]$. The w's that make this variance minimum must satisfy the equations obtained by replacing w_i with w_{n-i+1}, for $i > \frac{1}{2}(n + 1)$, in the expression

$$E[(w_1x_1 + w_2x_2 + \cdots + w_nx_n)^2] + \lambda(w_1 + w_2 + \cdots + w_n - 1)$$

and then setting the partial derivative with respect to each w equal to zero. This leads to

$$\sum_{j=1}^{n} w_j E[x_i x_j] + \sum_{j=1}^{n} w_j E[x_{n-i+1} x_j] + \lambda = 0, \qquad 1 \leq i \leq n,$$

where the summations include the terms $E[x_i^2]$ and $E[x_{n-i+1}{}^2]$, respectively. But it is known [2] that the sample mean is the regular unbiased estimate of μ with minimum variance. Setting each w equal to $1/n$ and combining equivalent terms yields

$$\sum_{j=1}^{n} E[x_i x_j] + \tfrac{1}{2}n\lambda = 0, \qquad i = 1, 2, \cdots, n.$$

Summing from $i = 1$ to $i = n$, and employing the relationships discussed in the preceding paragraph, we obtain

$$n + \tfrac{1}{2}n^2\lambda = 0,$$

whence

$$\lambda = -2/n,$$

and

$$\sum_{j=1}^{n} E[x_i x_j] = 1, \qquad i = 1, 2, \cdots, n,$$

where the summation includes the term $E[x_i^2]$. This equation leads to the properties mentioned at the beginning of this paragraph. The same equation can be used to evaluate $E[x_1^2]$ and $E[x_2^2]$ in samples of size 3 or 4 from the distribution $N(0, 1)$, after the product-moments have been found.

REFERENCES

[1] C. Hastings, Jr., F. Mosteller, J. W. Tukey, and C. P. Winsor, "Low moments for small samples; a comparative study of order statistics," *Annals of Math. Stat.*, Vol. 18 (1947), pp. 413–426.

[2] H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press, 1946, p. 483.

NOTE ON AN ASYMPTOTIC EXPANSION OF THE nTH DIFFERENCE OF ZERO

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This note gives an asymptotic expansion of the nth difference of zero. It is known that the Stirling number $S_{n,s}$ of the second kind is defined by

$$(1) \qquad n!\,S_{n,s} = \Delta^n 0^s = \sum_{x=0}^{n} (-1)^{n-x} \binom{n}{x} x^s.$$

We shall first show that the Stirling number $S_{n,n+k}$ can be expanded in the form

$$(2) \qquad S_{n,n+k} = \frac{n^{2k}}{2^k \cdot k!}\left[1 + \frac{f_1(k)}{n} + \frac{f_2(k)}{n^2} + \cdots + \frac{f_t(k)}{n^t} + O(n^{-t-1})\right], \quad (t < k)$$