MULTIPLE SAMPLING FOR VARIABLES

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Summary. A multiple (sequential) sampling scheme designed to test certain hypothesis is discussed with the following assumption: X is a random variable with density function $P(x)$ which is piecewise continuous and differentiable at its points of continuity. Formulae are derived for the probability of acceptance and rejection of the hypothesis and for the expected number of samples necessary for reaching a decision. These formulae are found to depend on the solution of a Fredholm Integral equation. Explicit solutions to the problem are obtained for the case when $P(x)$ is rectangular by reducing the fundamental integral equation to a set of differential-difference equations. Several examples are given.

1. Introduction. A multiple sampling scheme is here proposed which is based on cumulative sums of random variables. Bartky [1] has developed a theory of multiple sampling for attributes when the attribute can take only two values with probability (p) and $(1 - p)$ respectively. Formulae are there derived for the probabilities of acceptance and rejection of the null hypothesis and for the expected amount of sampling necessary for reaching a decision. In this paper the same type of formulae are developed for the case of variable sampling where the underlying probability law for the variable is given by a piecewise continuous function for which derivatives exist at its points of continuity.

The whole theory of multiple sampling is closely related to Wald's [2] theory of sequential tests. The fundamental difference is that in the latter, probabilities of errors of the first and second kinds are assigned, and acceptance and rejection criteria derived therefrom, while in the former the problem is solved in reverse order. There the acceptance and rejection criteria are assigned, and probabilities of eventual acceptance and rejection derived. For different parammeter values, these are the probabilities of making errors of the first and second kinds.

The problem presented here is similar to that given by Wald [3] in his paper on cumulative sums. In the present paper we waive the restriction that the expected number of items necessary for termination of the cumulating process be given explicitly as an integer. Since the theory given here is from the point of view of multiple sampling, the language appropriate to that theory will be used.

2. The sampling scheme. Let X be a random variable with probability density function $P(x)$ which is piecewise continuous. One variate, say x_1, is selected and if $x_1 > b$, the hypothesis (for example the null hypothesis with respect to the mean) is accepted, and if $x_1 < a$, the hypothesis is rejected. If, however, $a \leq x_1 \leq b$, another variate x_2 is selected. In the latter case similar criteria with respect to $x_1 + x_2$ determine whether the hypothesis is to be accepted or this method of sampling continued. Or more formally, let

$$S_r = \sum_{i=1}^{r} x_i \qquad (r = 1, 2, 3, \cdots),$$

where the cumulative sums S_r are formed sequentially as follows: for any integer r the cumulating process is terminated by acceptance of the hypothesis if $S_r > b$ and rejection if $S_r < a$, but, if $a \leq S_r \leq b$ another variate x_{r+1} is selected and the sum S_{r+1} formed. The acceptance and rejection criteria are then applied as above. No attempt is made here to indicate the choice of the acceptance and rejection criteria.

3. The probability of acceptance. If at the rth unit the hypothesis is neither accepted nor rejected, then it must be true that $a \leq S_r \leq b$. Let us denote the probability that this condition holds by

$$\int_a^b Y_r(S_r)\, dS_r, \tag{3.1}$$

where $Y_r(S_r)$ is the probability density function for S_r in the above described sampling scheme. The probability density function for S_{r+1} would then be given by

$$Y_{r+1}(S_{r+1}) = \int_a^b Y_r(S_r)P(S_{r+1} - S_r)\, dS_r. \tag{3.2}$$

The probabilities of accepting or rejecting the hypothesis on the rth trial are respectively

$$\int_b^{+\infty} Y_r(S_r)\, dS_r, \qquad \int_{-\infty}^{a} Y_r(S_r)\, dS_r, \tag{3.3}$$

and therefore the probabilities for eventual acceptance or rejection are given by

$$P_A = \sum_{r=1}^{\infty} \int_b^{\infty} Y_r(S_r)\, dS_r, \qquad P_R = \sum_{r=1}^{\infty} \int_{-\infty}^{a} Y_r(S_r)\, dS_r. \tag{3.4}$$

The probability that $a \leq S_n \leq b$ cannot exceed the probability that $a \leq T_n = x_1 + x_2 + x_3 + \cdots + x_n \leq b$ on a single sample of n variates, that is $\Pr(a \leq S_n \leq b) \leq \Pr(a \leq T_n \leq b)$. For distributions with positive variance, it can be shown that the right member of the above inequality approaches zero as $n \to 0$. Therefore, the process of sampling as outlined above will eventually lead to acceptance or rejection of the hypothesis. See Wald [3, p. 284] for a direct proof that the probability that the left member of the above inequality holds for $n = 1, 2, 3, \cdots$ is zero.

Consider the linear integral (Fredholm) equation

$$Y(x) = Y_1(x) + \lambda \int_a^b P(x - s) Y(s) \, ds, \tag{3.5}$$

where $Y_1(x) = P(x)$ and assume a solution of the form

$$Y(x) = Y_1(x) + \lambda Y_2(x) + \lambda^2 Y_3(x) + \cdots . \tag{3.6}$$

That solutions, in power series in λ, of the Fredholm equation exist when the kernel $P(x - s)$ and the function $Y_1(x)$ have finite discontinuities is well known and the theory has been expounded by several authors. (For example see Goursat [4].) If the power series in λ is substituted in the integral equation we obtain

$$\begin{aligned} Y_1(x) &+ \lambda Y_2(x) + \lambda^2 Y_3(x) + \cdots \\ &= Y_1(x) + \lambda \int_a^b [Y_1(s) + \lambda Y_2(s) + \lambda^2 Y_3(s) + \cdots] P(x - s) \, ds \\ &= Y_1(x) + \lambda \int_a^b Y_1(s) P(x - s) \, ds + \lambda^2 \int_a^b Y_2(s) P(x - s) \, ds \\ &\qquad + \lambda^3 \int_a^b Y_3(s) P(x - s) \, ds + \cdots . \end{aligned} \tag{3.7}$$

Equating coefficients of like powers of λ we see that

$$Y_r(x) = \int_a^b Y_{r-1}(s) P(x - s) \, ds, \qquad (r = 2, 3, \cdots). \tag{3.8}$$

This, however, is the probability distribution for S_r, $r = 2, 3, \cdots$ under our sampling scheme, and therefore from the equations,

$$Y(x) = \sum_{r=1}^{\infty} \lambda^{r-1} Y_r(x) = Y_1(x) + \lambda \int_a^b P(x - s) Y(s) \, ds, \tag{3.9}$$

we have that the probability of eventual acceptance for $\lambda = 1$, is

$$\sum_{r=1}^{\infty} \int_a^b Y_r(S_r) \, ds_r = \int_a^{\infty} Y(x) \, dx. \tag{3.10}$$

Thus our problem of finding a formula for the probability of eventual acceptance or rejection of the statistical hypothesis under the above sampling scheme reduces to that of finding a solution of a linear integral equation.

The argument in this section has, of course, been entirely formal. However from the general theory of integral equations we know that the series solution (3.6) converges uniformly for $\lambda < 1/M(b - a)$ where $P(x) \leq M$, since $P(x)$ is a probability density function. In equations (3.4) and (3.10) we give solutions for $\lambda = 1$ and of course we assume that $M(b - a) < 1$. Since (3.6) is uniformly convergent the interchanges of integration and summation in (3.10) and (4.3) in the following section are allowable.

4. The expected amount of sampling. Since

$$\int_a^b Y_{r-1}(S_{r-1})\, dS_{r-1} \tag{4.1}$$

is the probability that the rth sample will be reached, then the probability that on the rth sample, the hypothesis will be either accepted or rejected becomes

$$\int_a^b Y_{r-1}(S_{r-1})\, dS_{r-1} - \int_a^b Y_r(S_r)\, dS_r\,, \tag{4.2}$$

that is, the first term in this expression gives the probability that no terminating decision is made on the $(r - 1)$st sample and the second term gives the probability that a like decision is made on the rth sample. The difference of the two then gives the probability that a terminating decision (acceptance or rejection) will be made on the rth sample. The expected number of units sampled will therefore be

$$\begin{aligned} E &= 1 - \int_a^b P(x)\, dx + \sum_{r=2}^{\infty} r \left[\int_a^b Y_{r-1}(S_{r-1})\, dS_{r-1} - \int_a^b Y_r(S_r)\, dS_r \right] \\ &= 1 + \sum_{r=1}^{\infty} \int_a^b Y_r(S_r)\, dS_r = 1 + \int_a^b \sum_{r=1}^{\infty} Y_r(x)\, dx \\ &= 1 + \int_a^b Y(x)\, dx. \end{aligned} \tag{4.3}$$

Thus, the amount of sampling expected before a terminating decision is reached also depends upon the solution of the integral equation. We proceed to discuss the problem when $P(x)$ is given by a rectangular distribution.

5. Reduction to differential equations when P(x) is rectangular. Consider the integral equation

$$Y^*(z) = P^*(z) + \lambda \int_a^b P^*(z - t) Y^*(t)\, dt, \tag{5.1}$$

where

$$\begin{aligned} P^*(z) &= \frac{1}{2c}, \qquad -c \le z - \alpha \le +c; \\ &= 0, \qquad z - \alpha > c \quad \text{or} \quad z - \alpha < -c, \end{aligned} \tag{5.2}$$

and in the integral equation

$$a + \alpha - c < z < b + \alpha - c.$$

The parameter α is restricted to the values $-c \le \alpha \le c$ for the following reasons. The rejection criterion a cannot be greater than $c + \alpha$ for, if so, rejection will be automatic on the first sample. Similarly the acceptance criterion b must be greater than $-c + \alpha$ for otherwise, acceptance would be automatic on the first trial. If $\alpha > c$ then, rejection can never take place if it does not take place on the first trial for in this case all $z > 0$. Similarly, if $\alpha < -c$ then, acceptance can never take place if it does not take place on the first trial for in this case all $z < 0$. Furthermore, in obtaining solutions of the integral equation, we will take α to be ≥ 0. This inequality is no real restriction since solutions for negative α can be obtained by symmetry from the solutions for positive α.

If we let $x = z - \alpha$ then

$$Y^*(x + \alpha) = P^*(x + \alpha) + \lambda \int_a^b P^*(x + \alpha - t)Y^*(t)\, dt, \tag{5.3}$$

or

$$Y(x) = P(x) + \lambda \int_a^b P(x - t)Y^*(t)\, dt, \tag{5.4}$$

where

$$\begin{aligned} P(x) &= \frac{1}{2c}, \qquad -c \leq x \leq +c; \\ &= 0, \qquad x < -c \quad \text{or} \quad x > +c. \end{aligned} \tag{5.5}$$

Now let $s = t - \alpha$, then

$$\begin{aligned} Y(x) &= P(x) + \lambda \int_{a-\alpha}^{b-\alpha} P(x - \alpha - s)Y^*(s + \alpha)\, ds \\ &= P(x) + \lambda \int_{a-\alpha}^{b-\alpha} P(x - \alpha - s)Y(s)\, ds. \end{aligned} \tag{5.6}$$

We have thus transformed our equation to one in which $P(x)$ becomes symmetrical with respect to the line $x = 0$. Furthermore, the probability of acceptance becomes

$$P_A = \int_{b-\alpha}^{\infty} Y(x)\, dx, \tag{5.7}$$

and the expected amount of sampling becomes

$$E = 1 + \int_{a-\alpha}^{b-\alpha} Y(x)\, dx. \tag{5.8}$$

Also, x now has the following range: $a - c < x < b + c$. If we now make the transformation $x - \alpha - s = y$, then

$$Y(x) = P(x) + \int_{x-b}^{x-a} P(y)Y(x - \alpha - y)\, dy, \tag{5.9}$$

and the following cases present themselves.

If $x - a < -c$ or $x - b > +x$, then $Y(x) \equiv P(x)$, since $P(y) \equiv 0$.

If $x - b < -c < x - a < +c$, then

$$Y(x) = P(x) + \frac{\lambda}{2c}\int_{-c}^{x-a} Y(x - \alpha - y)\, dy, \tag{5.10}$$

where

$$\begin{aligned} &a - c \le x \le a + c \quad \text{when } b - a \ge 2c, \\ &a - c \le x \le b - c \quad \text{when } b - a \le 2c. \end{aligned} \tag{5.11}$$

If $x - b < -c < +c < x - a$, then

$$Y(x) = P(x) + \frac{\lambda}{2c}\int_{-c}^{+c} Y(x - \alpha - y)\, dy, \tag{5.12}$$

where

$$a + c \le x \le b - c \text{ and } b - c \ge 2c. \tag{5.13}$$

If $-c < x - b < x - a < +c$, then

$$Y(x) = P(x) + \frac{\lambda}{2c}\int_{x-b}^{x-a} Y(x - \alpha - y)\, dy, \tag{5.14}$$

where

$$b - c \le x \le a + c \text{ and } b - a \le 2c. \tag{5.15}$$

If $-c < x - b < +c < x - a$, then

$$Y(x) = P(x) + \frac{\lambda}{2c}\int_{x-b}^{+c} Y(x - \alpha - y)\, dy, \tag{5.16}$$

where

$$\begin{aligned} &b - c \le x \le b + c \text{ when } b - a \ge 2c, \\ &a + c \le x \le b + c \text{ when } b - a \le 2c. \end{aligned} \tag{5.17}$$

Transforming back to the variable s, we have for the case $b - a \ge 2c$,

$$\begin{aligned} Y(x) &= P(x) + \frac{\lambda}{2c}\int_{a-\alpha}^{x-\alpha+c} Y(s)\, ds \quad \text{for} \quad a - c \le x \le a + c, \\ &= P(x) + \frac{\lambda}{2c}\int_{x-\alpha-c}^{x-\alpha+c} Y(s)\, ds \quad \text{for} \quad a + c \le x \le b - c, \\ &= P(x) + \frac{\lambda}{2c}\int_{x-\alpha-c}^{b-\alpha} Y(s)\, ds \quad \text{for} \quad b - c \le x \le b + c, \end{aligned} \tag{5.18}$$

and for the case $b - a \le 2c$,

$$\begin{aligned} Y(x) &= P(x) + \frac{\lambda}{2c}\int_{a-\alpha}^{x-\alpha+c} Y(s)\, ds \quad \text{for} \quad a - c \le x \le b - c, \\ &= P(x) + \frac{\lambda}{2c}\int_{a-\alpha}^{b-\alpha} Y(s)\, ds \quad \text{for} \quad b - c \le x \le a + c, \\ &= P(x) + \frac{\lambda}{2c}\int_{x-\alpha-c}^{b-\alpha} Y(s)\, ds \quad \text{for} \quad a + c \le x \le b + c. \end{aligned} \tag{5.19}$$

In all of the above equations, the integral is a continuous function of x, α, a, b, c while $P(x)$ has a discontinuity at $x = +c$ and $x = -c$, the jump at these points being of amount $1/2c$. The function $Y(x)$ will therefore be such that

$$(5.20)\qquad \begin{aligned} Y(-c + 0) - Y(-c - 0) &= 1/2c, \\ Y(c - 0) - Y(c + 0) &= 1/2c. \end{aligned}$$

If we now differentiate the above sets of integral equations with respect to x we obtain the following sets of differential-difference equations for the case $\lambda = 1$. If $b - a \geq 2c$,

$$(5.21)\qquad \begin{aligned} Y'(x) &= \frac{1}{2c} Y(x - \alpha + c) && \text{for } a - c \leq x \leq a + c, \\ &= \frac{1}{2c} \{Y(x - \alpha + c) - Y(x - \alpha - c)\} && \text{for } a + c \leq x \leq b - c, \\ &= -\frac{1}{2c} Y(x - \alpha - c) && \text{for } b - c \leq x \leq b + c, \end{aligned}$$

and, if $b - a \leq 2c$,

$$(5.22)\qquad \begin{aligned} Y'(x) &= \frac{1}{2c} Y(x - \alpha + c) && \text{for } a - c \leq x \leq b - c, \\ &= 0 && \text{for } b - c \leq x \leq a + c, \\ &= -\frac{1}{2c} Y(x - \alpha - c) && \text{for } a + c \leq x \leq b + b, \end{aligned}$$

the derivatives holding for all points except at $x = -c$ and $x = +c$.

Although a technique has been devised to solve the above equations for finite a and b, mathematical difficulties of a computational character are encountered when $(b - a)$ is made large. Note that there are only three essential parameters in the above problem since c can be taken as the unit of measurement. In the technique illustrated by the following examples, α has been fixed as has $(b - a)$, i.e. the solutions shown in the examples below are general only insofar as one parameter is concerned. The essential feature of the technique is that the range of $Y(x)$ has been further subdivided so as to make its points of discontinuity end points of subdivisions of its range, and thus $Y(x)$ becomes continuous from the right or left in every subinterval of its range.

6. Example I: $b - a = 2c$, $c = 1$, $\alpha = 0$. In this case x ranges from $(a - 1)$ or $(-c)$, whichever is smaller, to $(b + 1)$ or $(+c)$, whichever is larger. If $-c < a - 1$, then $Y(x) = P(x)$ for $-c \leq x < a - 1$, and if $b + 1 < +c$, then $Y(x) = P(x)$ for $b + 1 < x < -c$. For x between $a - 1$ and $b + 1$ the domain of the differential-difference equations is divided as follows, where a is now restricted to the interval $-1 \leqq a \leqq 0$.

$$
\begin{aligned}
(6.1)\quad Y_i'(x) &= \tfrac{1}{2}Y_{i+2}(x+1) \text{ where for } & i&=1, & a-1 &\le x < -1,\\
&& i&=2, & -1 &\le x < a,\\
&& i&=3, & a &\le x < 0,\\
&& i&=4, & 0 &\le x < a+1;\\
Y_i'(x) &= -\tfrac{1}{2}Y_{i-2}(x-1) \text{ where for } & i&=5, & a+1 &\le x < +1,\\
&& i&=6, & +1 &\le x < a+2,\\
&& i&=7, & a+2 &\le x < +2,\\
&& i&=8, & +2 &\le x < a+3.
\end{aligned}
$$

The above are the equations corresponding to (5.21) for the given example.

Differentiating the equations for $i = 3, 4, 5, 6$ and making certain obvious substitutions we obtain the following second order differential equations,

$$(6.2)\qquad Y_i''(x) = -\tfrac{1}{4}Y_i(x), \qquad i = 3, 4, 5, 6,$$

where the domains for x are as in (6.1). If we solve the equations (6.2) and substitute in the remaining equations in (6.1) we obtain the following set of equations,

$$
(6.3)\qquad
\begin{aligned}
Y_i(x) &= A_{i+2}\sin\tfrac{1}{2}(x+1) - B_{i+2}\cos\tfrac{1}{2}(x+2) + K_i, & i&=1,2,\\
Y_i(x) &= A_i\cos\tfrac{1}{2}x + B_i\sin\tfrac{1}{2}x, & i&=3,4,5,6,\\
Y_i(x) &= -A_{i-2}\sin\tfrac{1}{2}(x-1) + B_{i-2}\cos\tfrac{1}{2}(x-1) + K_i, & i&=7,8,
\end{aligned}
$$

where again the domains are as in (6.1)

From continuity considerations we have the boundary conditions

$$
\begin{gathered}
Y_1(a-1) = Y_8(a+3) = 0, \qquad Y_1(-1) - \tfrac{1}{2} = Y_2(-1), \qquad Y_2(a) = Y_3(a),\\
Y_3(0) = Y_4(0), \qquad Y_4(a+1) = Y_5(a+1), \qquad Y_5(1) = Y_6(1) + \tfrac{1}{2},\\
Y_6(a+2) = Y_7(a+2), \qquad Y_7(2) = Y_8(2).
\end{gathered}
$$

These boundary conditions yield certain relationships between the constants. The equations so determined, however, do not form a consistent set of linear equations in the A_i, B_i, $K_i \cdots$. If we integrate out the equations (5.18), sectionally, the following relationships between the constants are obtained.

$$
(6.4)\qquad
\begin{aligned}
&A_i = A_{i+2}\sin\tfrac{1}{2} - B_{i+2}\cos\tfrac{1}{2}, \qquad B_i = B_{i+2}\cos\tfrac{1}{2} + B_{i+2}\sin\tfrac{1}{2}, \qquad i = 3,4,\\
&K_2 = -(A_4 - A_5)\sin\tfrac{1}{2}(a+1) - (B_5 - B_4)\cos\tfrac{1}{2}(a+1)\\
&\qquad\qquad = \tfrac{1}{2} + B_4 - B_3 + K_1,\\
&K_7 = A_3 - A_4 + K_8, \qquad K_8 = A_6\sin\tfrac{1}{2}(a+2) - B_6\cos\tfrac{1}{2}(a+2),\\
&B_3 = \tfrac{1}{2} + B_4 + K_1 - A_4 + A_3 + (A_4 - A_5)\sin\tfrac{1}{2}(a+1)\\
&\qquad\qquad + (B_5 - B_4)\cos\tfrac{1}{2}(a+1),\\
&A_4 = A_3 + K_8 + (A_4 - A_5)\sin\tfrac{1}{2}(a+1) + (B_5 - B_4)\cos\tfrac{1}{2}(a+1),\\
&K_1 = -A_3\sin\frac{a}{2} + B_3\cos\frac{a}{2}.
\end{aligned}
$$

From these equations it is easily seen that $A_4 = A_3$ and $K_1 = K_2 = K_7 = K_8$. Furthermore, the following set of consistent linear equations is obtained, after several simple manipulations and substitutions.

$$
(6.5)\quad
\begin{aligned}
&\left\{\sin \tfrac{1}{2}(a+2) + \sin \frac{a}{2} \cdot \sin \tfrac{1}{2}\right\} A_6 \\
&\qquad - \left\{\cos \frac{a}{2}\right\} B_3 + \left\{\cos \tfrac{1}{2}(a+2) + \sin \frac{a}{2} \cos \tfrac{1}{2}\right\} B_6 = 0, \\
&\left\{- \sin \tfrac{1}{2}(a+2) + \cos \tfrac{1}{2}(a+2) - \cos \frac{a}{2} \cdot \sin \tfrac{1}{2}\right\} A_6 + \left\{\sin \frac{a}{2}\right\} B_3 \\
&\qquad + \left\{\sin \tfrac{1}{2}(a+2) + \cos \tfrac{1}{2}(a+2) - \cos \frac{a}{2} \cdot \cos \tfrac{1}{2}\right\} B_6 = 0, \\
&\qquad \{\cos \tfrac{1}{2}\} A_6 - B_3 + \{\sin \tfrac{1}{2}\} B_6 = 0.
\end{aligned}
$$

All the other constants can be obtained from the solutions for A_6, B_3, B_6 in (6.5). Letting Δ equal the determinant of coefficients in (6.5) and using the relationships (6.4) we obtain the following solutions:

$$
(6.6)\quad
\begin{aligned}
\Delta &= 2 - 2 \sin \tfrac{1}{2} - \cos \tfrac{1}{2}, \\
\Delta A_4 &= \tfrac{1}{2}\{\cos \tfrac{1}{2} - \cos a/2 \cdot \sin \tfrac{1}{2}(a+1)\} = \Delta A_3, \\
\Delta B_4 &= \tfrac{1}{2}\{\sin \tfrac{1}{2} - \sin a/2 \cdot \sin \tfrac{1}{2}(a+1) + \cos \tfrac{1}{2} - 1\}, \\
\Delta A_6 &= \tfrac{1}{2}\{\sin 1 - \cos \tfrac{1}{2} + \cos a/2 \cdot \cos \tfrac{1}{2}(a+2)\}, \\
\Delta B_6 &= \tfrac{1}{2}\{\sin \tfrac{1}{2}(a+2) \cos a/2 - \sin \tfrac{1}{2} - \cos 1\}, \\
\Delta B_3 &= \tfrac{1}{2}\{1 - \sin a/2 \cdot \sin \tfrac{1}{2}(a+1) - \sin \tfrac{1}{2}\}, \\
\Delta A_5 &= \tfrac{1}{2}\{\cos \tfrac{1}{2} - \sin^2 \tfrac{1}{2}(a+1)\}, \\
\Delta B_5 &= \tfrac{1}{2}\{\sin \tfrac{1}{2} + \sin \tfrac{1}{2}(a+1) \cdot \cos \tfrac{1}{2}(a+1) - 1\}, \\
\Delta K_1 &= \tfrac{1}{2}\left\{\cos \frac{a}{2} \sin \tfrac{1}{2}(a+2)\right\} = \Delta K_2 = \Delta K_7 = \Delta K_8.
\end{aligned}
$$

If we now integrate $Y(x)$, equation (6.3) sectionally, i.e. from the left end point to the right end point of each sub-interval of its range and then add up appropriate areas, we obtain the following formulae for the probabilities of acceptance and rejection and for the expected amount of sampling:

$$
(6.7)\quad
\begin{aligned}
P_R &= \frac{1}{\Delta}\{\cos \tfrac{1}{2}(a+1) + \sin a/2 - \cos a/2 + \Delta K_2\}, \\
P_A &= \frac{1}{\Delta}\{2 - \cos \tfrac{1}{2} - 2 \sin \tfrac{1}{2} + \sin \tfrac{1}{2}(a+1) \\
&\qquad - \cos \tfrac{1}{2}(a+1) - \sin a/2 + \Delta K_2\}, \\
E &= \frac{1}{\Delta}\{\cos a/2 - 2 \sin a/2 - \sin \tfrac{1}{2}(a+1)\}.
\end{aligned}
$$

7. Example II: $\alpha = 1, c = 3, b - a = 4$. In this case, as in the previous one, $Y(x) = P(x)$ for $-3 \le x < a - 3$ when $a - c = a - 3 < -3$ and if $b + c = a + 7 < 3$ then $Y(x) = P(x)$, $a + 7 \le x < 3$. For $a - 3 \le x \le a + 7$ where a takes on only integral values between -5 and 3, we have the following set of differential-difference equations:

$$
(7.1) \qquad \begin{aligned} Y'_{a+j}(x) &= \tfrac{1}{6} Y_{a+j+2}(x+2), && j = -3, -2, -1, 0; \\ &\doteq 0, && j = 1, 2; \\ &= -\tfrac{1}{6} Y_{a+j-4}(x-4), && j = 3, 4, 5, 6. \end{aligned}
$$

If we integrate the above equations for $j = 1, 2$, substitute in the equations for $j = -1, 0, 5, 6$, integrate, and then substitute in the remaining equations, we obtain the solutions

$$
(7.2) \qquad \begin{aligned} Y_{a+j}(x) &= \tfrac{1}{72} A_{a+j+4}(x+2)^2 + \tfrac{1}{6} A_{a+j+2}\, x + A_{a-j}, && j = -3, -2; \\ &= \tfrac{1}{6} A_{a+j+2}\, x + A_{a+j} && j = -1, 0; \\ &= A_{a+j} && j = 1, 2; \\ &= -\tfrac{1}{72} A_{a+j-2}(x-4)^2 - \tfrac{1}{6} A_{a+j-4}\, x + A_{a+j}, && j = 3, 4; \\ &= -\tfrac{1}{6} A_{a+j-4}\, x + A_{a+j}, && j = 5, 6. \end{aligned}
$$

As in the previous example we now use (5.22). Integrating out (5.22) sectionally, certain relationships between the A_{a+j}, $j = -3, -2, \cdots 6$, are obtained. These yield

$$
(7.3) \qquad \begin{aligned} A_{a+1} &= \tfrac{1}{28}\{12P_{a-1} + 12P_a + 39P_{a+1} + 9P_{a+2}\}, \\ A_{a+2} &= \tfrac{1}{28}\{12P_{a-1} + 12P_a + 11P_{a+1} + 37P_{a+2}\}, \\ A_{a-1} &= -\frac{a}{56}\{4P_{a-1} + 4P_a + 13P_{a+1} + 3P_{a+2}\} \\ &\qquad + \tfrac{1}{168}\{228P_{a-1} + 60P_a + 55P_{a+1} + 17P_{a+2}\}, \\ A_a &= -\frac{a}{168}\{12P_{a-1} + 12P_a + 11P_{a+1} + 37P_{a+2}\} \\ &\qquad + \tfrac{1}{168}\{60P_{a-1} + 228P_a + 55P_{a+1} + 17P_{a+2}\}, \end{aligned}
$$

where P_{a+j} is the value of $P(x)$ for $a + j \le x \le a + j + 1$, $j = -3, \cdots 6$. All of the other constants can be found in terms of those given in equations (7.3). If we now integrate (7.2) sectionally and perform several simple manipulations, we arrive at the following formulas:

$$
(7.4) \qquad \begin{aligned} P_R &= \sum_{j=-6}^{-2} P_{a+j} + \frac{9a-5}{216} A_{a+1} + \frac{3a-1}{216} A_{a+2} + \frac{3}{12} A_{a-1} - \frac{1}{12} A_a, \\ P_A &= \sum_{j=3}^{9} P_{a+j} + \frac{3a-89}{216} A_{a+1} + \frac{9a-131}{216} A_{a+2} + \tfrac{1}{12} A_{a-1} + \tfrac{3}{12} A_a, \\ E &= 1 + \frac{2a-11}{12} A_{a+1} + \frac{2a-13}{12} A_{a+2} + A_{a+1} + A_a. \end{aligned}
$$

Although P_{a+j}, $j = -6, -5, -4, 7, 8, 9$, have not appeared in previous derivations in this example, they have been inserted in the above formulas to cover the cases in which $a - c > -c$ or $b + c < c$.

It should be mentioned that Kac [5] obtained the distribution of n (the expected amount of sampling) by a process similar to that given in this paper. It is also interesting to note that the present paper could have been written entirely in the language of problems in Random Walk.

The author has also worked on the case in which the distribution $P(x)$ is triangular and parabolic. In these, as in the case of the rectangular distribution discussed in this paper for $b - a$ large, the equivalent differential-difference equations are of large orders making the computation of solutions extremely tedious. As Kac [5] pointed out, the task of obtaining solutions in closed form for the case when $P(x)$ is the normal law is extremely difficult.

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