INVERSION FORMULAE FOR THE DISTRIBUTION OF RATIOS

BY JOHN GURLAND

University of California, Berkeley

1. Summary. The use of the repeated Cauchy principal value affords greater facility in the application of inversion formulae involving characteristic functions. Formula (2) below is especially useful in obtaining the inversion formula (1) for the distribution of the ratio of linear combinations of random variables which may be correlated. Formulae (1), (10), (12) generalize the special cases considered by Cramer [2], Curtiss [4], Geary [6], and are free of some restrictions they impose. The results are further generalized in section 6, where inversion formulae are given for the joint distribution of several ratios. In section 7, the joint distribution of several ratios of quadratic forms in random variables $X_1, X_2, \cdots, X_n$ having a multivariate normal distribution is considered.

2. Introduction. We shall write

$$\oint \oint \cdots \oint g(t_1, t_2, \cdots, t_n)\, dt_1\, dt_2 \cdots dt_n$$

$$= \lim_{\substack{\epsilon_i \to 0 \\ T_i \to \infty}} \int\int_{\epsilon_i < |t_i| < T_i} \cdots \int g(t_1, t_2, \cdots, t_n)\, dt_1\, dt_2 \cdots dt_n,$$

which might be called the repeated Cauchy principal value of

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(t_1, t_2, \cdots, t_n)\, dt_1 \cdots dt_n,$$

and which we shall use frequently. The results of this article may be regarded as extensions of the following theorem proved in section 4.

THEOREM 1. *Let $X_1, X_2, \cdots, X_n$ have the joint distribution function $F(x_1, x_2, \cdots, x_n)$ with corresponding characteristic function $\phi(t_1, t_2, \cdots, t_n)$. Let $G(x)$ be the distribution function of $(a_1X_1 + \cdots + a_nX_n)/(b_1X_1 + \cdots + b_nX_n)$, where $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are real numbers. If*

$$P\left\{\sum_1^n b_j x_j \leq 0\right\} = 0,$$

then

$$G(x) + G(x - 0) = 1 - \frac{1}{\pi i} \oint \frac{\varphi\{t(a_1 - b_1 x), \cdots, t(a_n - b_n x)\}}{t}\, dt, \tag{1}$$

3. An inversion formula for distribution functions. Let $F(x)$ be a distribution function and $\phi(t)$ be the corresponding characteristic function. Then the following inversion formula holds:

$$F(\xi) + F(\xi - 0) = 1 - \frac{1}{\pi i} \oint e^{-it\xi} \phi(t) \frac{dt}{t}. \tag{2}$$

PROOF.

$$\left(\int_{-T}^{-\epsilon} + \int_{\epsilon}^{T}\right) e^{-it\xi} \phi(t) \frac{dt}{t} = \left(\int_{-T}^{-\epsilon} + \int_{\epsilon}^{T}\right) \frac{e^{-it\xi}}{t} dt \int_{-\infty}^{\infty} e^{itx}\, dF(x)$$
$$= \int_{-\infty}^{\infty} dF(x) \left(\int_{-T}^{-\epsilon} + \int_{\epsilon}^{T}\right) e^{it(x-\xi)} \frac{dt}{t},$$

by the Fubini theorem on the inversion of integrals. But

$$\frac{1}{\pi i} \oint e^{it(x-\xi)} \frac{dt}{t} = \operatorname{sgn}(x - \xi),$$

where $\operatorname{sgn} y = -1, 0, 1$ according as $y < 0, y = 0, y > 0$. Since $\int_{-T}^{T} \frac{\sin at}{t} dt$ is uniformly bounded in T, the principle of bounded convergence for Lebesgue integrals implies that

$$\frac{1}{\pi i} \lim_{\substack{\epsilon \to 0 \\ T \to \infty}} \int_{-\infty}^{\infty} dF(x) \left(\int_{-T}^{-\epsilon} + \int_{\epsilon}^{T}\right) e^{it(x-\xi)} \frac{dt}{t} = \int_{-\infty}^{\infty} \operatorname{sgn}(x - \xi)\, dF(x)$$
$$= \left(\int_{-\infty}^{\xi - 0} + \int_{(\xi)} + \int_{\xi+0}^{\infty}\right) \operatorname{sgn}(x - \xi)\, dF(x)$$
$$= -F(\xi - 0) + 1 - F(\xi).$$

The required result follows at once.

Another form of (2) may be obtained as follows: Let $H(x)$, $K(x)$ be distribution functions, and $\psi(t)$, $\chi(t)$ the corresponding characteristic functions. Setting $F = H, \phi = \psi, \xi = 0$ in (2) yields

$$H(0) + H(0 - 0) = 1 - \frac{1}{\pi i} \oint \psi(t) \frac{dt}{t},$$

while setting $F = K, \phi = \chi$, in (2) yields

$$K(\xi) + K(\xi - 0) = 1 - \frac{1}{\pi i} \oint \chi(t) e^{-it\xi} \frac{dt}{t}.$$

Clearly

$$K(\xi) + K(\xi - 0) = H(0) + H(0 - 0) + \frac{1}{\pi i} \oint \frac{\psi(t) - \chi(t) e^{-it\xi}}{t} dt. \tag{3}$$

If $H = K$, then $\psi = \chi$, and (3) reduces to a well-known inversion formula (cf. Kendall [7, p. 91]).

4. Distribution of the ratio $(a_1 X_1 + \cdots + a_n X_n)/(b_1 X_1 + \cdots + b_n X_n)$ with denominator positive. THEOREM 1. *Let $X_1, X_2, \cdots, X_n$ have the joint* *distribution function* $F(x_1, x_2, \cdots, x_n)$ *with corresponding characteristic function* $\phi(t_1, t_2, \cdots, t_n)$. *Let* $G(x)$ *be the distribution function of* $(a_1X_1 + \cdots + a_nX_n)/(b_1X_1 + \cdots + b_nX_n)$ *where* $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ *are real numbers. If* $P\{\Sigma_1^n b_jX_j \leq 0\} = 0$, *then*

$$G(x) + G(x-0) = 1 - \frac{1}{\pi i} \oint \frac{\phi\{t(a_1 - b_1x), \cdots, t(a_n - b_nx)\}}{t}\, dt.$$

PROOF. Note that

$$P\left\{\frac{\Sigma a_iX_i}{\Sigma b_iX_i} \leq x\right\} = P\{\Sigma(a_i - b_ix)X_i \leq 0\},$$

and let $R_x(\xi) = P\{\Sigma(a_i - b_ix)X_i \leq \xi\}$ and $\chi_x(t)$ be the corresponding characteristic function. Clearly $R_x(0) = G(x)$ and

$$\chi_x(t) = \phi\{t(a_1 - b_1x), \cdots, t(a_n - b_nx)\}.$$

On applying (2) to $R_x(\xi)$ and setting $\xi = 0$, the required result follows at once. If (3) is applied in place of (2), with $K = G$, we obtain

$$G(x) + G(x-0) = H(0) + H(0-0) + \frac{1}{\pi i} \oint \frac{\psi(t) - \phi\{t(a_1 - b_1x), \cdots, t(a_n - b_nx\}}{t}\, dt. \tag{4}$$

We shall consider (3) and (4) when $n = 2$ and

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Two cases will be treated separately; first, when X_1, X_2 are independent, second, when X_1, X_2 may be correlated.

If X_1, X_2 are independent, and $F(x_1, \infty) = F_1(x_1)$, $F(\infty, x_2) = F_2(x_2)$, with corresponding characteristic functions $\phi_1(t)$, $\phi_2(t)$ then (1) becomes

$$G(x) + G(x-0) = 1 - \frac{1}{\pi i} \oint \frac{\phi_1(t)\phi_2(-tx)}{t}\, dt; \tag{5}$$

while (4) becomes, taking $H = F$

$$G(x) + G(x-0) = \frac{1}{\pi i} \oint \frac{\phi_2(t) - \phi_1(t)\phi_2(-tx)}{t}\, dt. \tag{6}$$

Cramér [2, p. 46] proves, for X_1, X_2 independent and $F_2(0) = 0$, that

$$G(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi_2(t) - \phi_1(t)\phi_2(-tx)}{t}\, dt$$

under the following conditions:

(i) X_1 and X_2 have finite means;

(ii) $\int_1^\infty \left|\frac{\phi_2(t)}{t}\right| dt < \infty$.

If X_1, X_2 may be correlated, then (1) becomes

$$G(x) + G(x - 0) = 1 - \frac{1}{\pi i} \oint \frac{\phi(t, -tx)}{t}\, dt; \tag{7}$$

while (4) becomes, taking $H = F$,

$$G(x) + G(x - 0) = \frac{1}{\pi i} \oint \frac{\phi_2(t) - \phi(t, -tx)}{t}\, dt. \tag{8}$$

Professor P. L. Hsu, in a course of lectures attended by the author at the Statistical Laboratory, University of California, gave the following result of Cramér, which was stated thus, using the above notation:

$$G(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi_2(t) - \chi_x(t)}{t}\, dt, \tag{9}$$

provided $\int_{-\infty}^{\infty} \left| \frac{\phi_2(t) - \chi_x(t)}{t} \right| dt < \infty$, where $F_2(0) = 0$ and $\chi_x(t)$ is defined above expression (4).

The following corollary is obtained from (1) according to well-known theorems concerning differentiation under the integral sign:

COROLLARY. *Suppose $\phi(t_1, t_2, \cdots, t_n)$ is the characteristic function corresponding to $X_1, X_2, \cdots, X_n$, and $G(x)$ is the distribution function of*

$$(a_1X_1 + \cdots + a_nX_n)/(b_1X_1 + \cdots + b_nX_n);$$

then, if $P\{\Sigma b_i X_i \leq 0\} = 0$,

$$G'(x) = \frac{1}{2\pi i} \oint \left[\sum_{k=1}^{n} b_k \frac{\partial \phi(t_1, \cdots, t_n)}{\partial t_k} \right]_{t_k = t(a_k - b_k x)} dt, \tag{10}$$

in every interval in which the integral converges uniformly.

If $n = 2$, and

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$G'(x) = \frac{1}{2\pi i} \oint \left[\frac{\partial \phi(t_1, t_2)}{\partial t_2} \right]_{t_2 = -t_1 x} dt_1. \tag{11}$$

Cramér [3, p. 317, exercise 6] states the following result:

If $F(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(u, v)\, du\, dv$, and $F_2(0) = 0$, then

$$G'(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[\frac{\partial \phi(t_1, t_2)}{\partial t_2} \right]_{t_2 = -t_1 x} dt_1,$$

if the integral is uniformly convergent with respect to x.

Geary [6] has shown that if $F(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(u, v)\, du\, dv$, $F_2(0) = 0$, and $\lambda(t, v) = \int_{-\infty}^{\infty} e^{itv} f(u, v)\, du$, then

$$G'(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[\frac{\partial \phi(t_1, t_2)}{\partial t_2} \right]_{t_2 = -t_1 x} dt_1,$$

provided

(i) $$\phi(t_1\ t_2) = 0 \text{ for } t_2 = \pm\infty,$$

(ii) $$\int_0^{\infty} dy \int_{-\infty}^{\infty} y\lambda(t, y) e^{-ityx}\, dt = \int_{-\infty}^{\infty} dt \int_0^{\infty} y\lambda(t, y) e^{-ityx}\, dy.$$

Formula (1) can be employed in the case $n = 2$, X_1, X_2 are independent, and

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

to obtain closed expressions for the distribution functions of ratios in which the variable in the numerator and that in the denominator may have any one of the following four distributions: Binomial, Rectangular, χ^2, Normal. In the case of the four ratios with the binomially distributed variable as the denominator, a translation must be made to ensure positiveness of the denominator. For the four ratios with the normally distributed variable as denominator, the distribution function obtained is approximate; and the approximation is good if $P\{X_2 \leq 0\}$ is sufficiently small (cf. Geary [5]).

5. Distribution of the ratio $(a_1 X_1 + \cdots + a_n X_n)/(b_1 X_1 + \cdots + b_n X_n)$, with denominator positive or negative. The following theorem will be proven:

THEOREM 2. *Let $G(x)$ be the distribution function of $(a_1X_1 + \cdots + a_nX_n)/(b_1X_1 + \cdots + b_nX_n)$ where $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are real numbers. If $P\{\Sigma_1^n b_1X_1 = 0\} = 0$, then*

$$G(x) + G(x - 0) = 1 - \frac{1}{\pi i} \oint \frac{\phi^+\{t(a_1 - b_1 x), \cdots, t(a_n - b_n x)\} + \phi^-\{t(a_1 - b_1 x), \cdots, t(a_n - b_n x)\}}{t}\, dt, \tag{12}$$

where

$$\phi^+(t_1, t_2, \cdots, t_n) = \underset{\Sigma b_k x_k > 0}{\iint \cdots \int} e^{i(t_1x_1 + \cdots + t_nx_n)}\, dF(x_1, x_2, \cdots, x_n),$$

$$\phi^-(t_1, t_2, \cdots, t_n) = \underset{\Sigma b_k x_k < 0}{\iint \cdots \int} e^{i(t_1x_1 + \cdots + t_nx_n)}\, dF(x_1, x_2, \cdots, x_n).$$

PROOF. Let $R_x(\xi) = P\{\Sigma b_k X_k > 0\} \cdot P\{\Sigma X_k(a_k - b_k x) \leq \xi \mid \Sigma b_k X_k > 0\}$
$+ P\{\Sigma b_k X_k < 0\} \cdot P\{\Sigma X_k(a_k - b_k x \geq -\xi \mid \Sigma b_k X_k < 0\}$.

Then $R_x(\infty) = 1$, $R_x(-\infty) = 0$, and $R_x(\xi)$ is non-decreasing in ξ and continuous on the right. Hence $R_x(\xi)$ is a distribution function (Cramér [2, p. 11]). It can be shown by a proof analogous to that used by Curtiss [4] that the characteristic function of $R_x(\xi)$ is

$$\phi^{+}\{t(a_1 - b_1 x), \cdots, t(a_n - b_n x)\} + \phi^{-}\{t(b_1 x - a_1), \cdots, t(b_n x - a_n)\}$$

Since $R_x(0) = G(x)$, application of (2) to $R_x(\xi)$ yields the required result.

6. Inversion formulae for multidimensional distribution functions. The n-dimensional analogue of (2) will now be given, and will be applied to obtain inversion formulae for the joint distribution of several ratios.

Let $X_1, X_2, \cdots, X_n$ have the joint distribution function $F(x_1, x_2, \cdots, x_n)$ and the corresponding characteristic function $\phi(t_1, t_2, \cdots, t_n)$. Let

$$\phi_{j_1, j_2, \cdots, j_k}(t_1, t_2, \cdots, t_k)$$

be the characteristic function corresponding to the marginal joint distribution function of $X_{j_1}, X_{j_2}, \cdots, X_{j_k}$, where the set $j_1, j_2, \cdots, j_k$ is a permutation of k of the integers $1, 2, \cdots, n$. Note that

$$\phi(t_1, t_2, \cdots, t_n) = \phi_{1, 2, \cdots, n}(t_1, t_2, \cdots, t_n).$$

The summation $\sum_{(i_1, i_2, \cdots, i_n)} F(\xi_{1i_1}, \xi_{2i_2}, \cdots, \xi_{ni_n})$, which will appear below is to be interpreted as follows:

Defining $\xi_{ji_j} = \xi_j$ if $i_j = 1$,
$= \xi_j - 0$ if $i_j = 0$,

then $\sum_{(i_1, i_2, \cdots, i_n)} F(\xi_{1i_1}, \cdots, \xi_{ni_n})$ will mean that the summation is to be taken over all binary numbers $.i_1 i_2 \cdots i_n$.

Using the notation of the preceding paragraph, we can state the following theorem:

THEOREM 3. *Let $A_0, A_1, \cdots, A_n$ satisfy the $n + 1$ equations*

$$\sum_{k=0}^{n-r-1} \binom{n-r}{k} A_{r+k} = 1, \qquad A_n = -1, \qquad (r = 0, 1, 2, \cdots, n-1),$$

where $\binom{m}{p}$ as usual, denotes the binomial coefficient.

Then

$$(13) \quad (-1)^{n+1} \sum_{(i_1, i_2, \cdots, i_n)} F(\xi_{1i_1}, \cdots, \xi_{ni_n}) = A_0 + \sum_{k=1}^{n} \frac{A_k}{(\pi i)^k} \sum_{j_1 < j_2 < \cdots < j_k} \oint \oint \cdots \oint \cdot \exp\{-i(t_1 \xi_{j_1} + \cdots + t_k \xi_{j_k})\} \phi_{j_1 j_2 \cdots j_k}(t_1, t_2, \cdots, t_k) \frac{dt_1 \, dt_2 \cdots dt_k}{t_1 t_2 \cdots t_k}.$$

PROOF: Since the theorem is already proved for $n = 1$ (section 3), and since

$$\frac{1}{(\pi i)^n} \oint\oint \cdots \oint e^{-i(t_1\xi_1+\cdots+t_n\xi_n)} \phi(t_1, t_2, \cdots, t_n) \frac{dt_1\, dt_2 \cdots dt_n}{t_1 t_2 \cdots t_n}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \operatorname{sgn}(x_1 - \xi_1) \operatorname{sgn}(x_2 - \xi_2) \cdots$$
$$\cdots \operatorname{sgn}(x_n - \xi_n)\, dF(x_1, x_2, \cdots, x_n),$$

the theorem could be proven by induction. The result is obtained more quickly, however, by noting that it suffices to consider the case of independent $X_1, X_2, \cdots, X_n$.

It may be remarked that if $(\xi_1, \xi_2, \cdots, \xi_n)$ is a continuity point of $F(x_1, x_2, \cdots, x_n)$, the left-hand member of (13) becomes

$$(-1)^{n+1}2^n F(\xi_1, \xi_2, \cdots, \xi_n),$$

and also that differentiation of (13) yields

$$\frac{\partial^n F(\xi_1, \xi_2, \cdots, \xi_n)}{\partial\xi_1\, \partial\xi_2 \cdots \partial\xi_n} = \left(\frac{1}{2\pi}\right)^n \oint\oint \cdots \oint e^{-i(t_1\xi_1+\cdots+t_n\xi_n)} \phi(t_1, t_2, \cdots, t_n)\, dt_1\, dt_2 \cdots dt_n, \tag{14}$$

in every n-dimensional interval in which the integral converges uniformly. This agrees with well-known results concerning Fourier inversion formulae.

An inversion formula for the joint distribution of p ratios

$$\frac{a_{1i}X_1 + a_{2i}X_2 + \cdots + a_{ni}X_n}{b_{1i}X_1 + b_{2i}X_2 + \cdots + b_{ni}X_n}; \qquad i = 1, 2, \cdots, p \qquad (1 \leq p \leq n),$$

can be obtained from (13) by a method similar to that applied in section 4. The following theorem holds:

THEOREM 4. *Let*

$$G(\xi_1, \xi_2, \cdots, \xi_p) = P\left\{\frac{\Sigma a_{j1}X_j}{\Sigma b_{j1}X_j} \leq \xi_1; \cdots; \frac{\Sigma a_{jp}X_j}{\Sigma b_{jp}X_j} \leq \xi_p\right\}$$

and $\phi(t_1, t_2, \cdots, t_n)$ be the characteristic function corresponding to X_1, $X_2, \cdots, X_n$. Then, if $P\{\Sigma b_{jk}X_j \leq 0\} = 0$ $(k = 1, 2, \cdots p)$,

$$(-1)^{p+1} \sum_{(i_1 i_2 \cdots i_p)} G(\xi_{1i_1}, \xi_{2i_2}, \cdots, \xi_{pi_p})$$
$$= A_0 + \sum_{k=1}^{p} \frac{A_k}{(\pi i)^k} \sum_{i_1<i_2<\cdots<i_k} \oint\oint \cdots \oint \phi\left\{\sum_{l=1}^{k} t_l(a_{1j_l} - b_{1j_l}\xi_{j_l}), \right. \tag{15}$$
$$\left. \cdots, \sum_{l=1}^{k} t_l(a_{nj_l} - b_{nj_l}\xi_{j_l})\right\} \frac{dt_1\, dt_2 \cdots dt_k}{t_1 t_2 \cdots t_k}.$$

The following corollary is a generalization of (10) and follows by differentiation of (15):

COROLLARY. *Suppose* $G(x_1, x_2, \cdots, x_p)$ *is the joint distribution function of the* p *ratios*

$$\frac{a_{1j}X_1 + \cdots + a_{nj}X_n}{b_{1j}X_1 + \cdots + b_{nj}X_n},$$

and $\phi(t_1, t_2, \cdots, t_n)$ *is the characteristic function corresponding to* $X_1, X_2, \cdots, X_n$, *then*, if $P\{\Sigma_{i=1}^n b_{ij}X_i \leq 0\} = 0, j = 1, 2, \cdots p$.

$$\text{(16)} \quad \begin{aligned} \frac{\partial^p G(\xi_1 \xi_2 \cdots \xi_p)}{\partial \xi_1 \partial \xi_2 \cdots \partial \xi_p} &= \left(\frac{1}{2\pi i}\right)^p \oint \oint \cdots \oint \\ &\cdot \left[\sum_{k=1}^n b_{k1} b_{k2} \cdots b_{kp} \frac{\partial^p \phi(t_1, t_2, \cdots t_n)}{\partial t_k^p}\right]_{t_k = \sum_{i=1}^p \tau_j (a_{kj} - b_{kj}\xi_j)} d\tau_1 \, d\tau_2 \cdots d\tau_p, \end{aligned}$$

in every p-dimensional interval in which the integral converges uniformly.

7. Joint distribution of ratios of quadratic forms. *Let* $X_1, X_2, \cdots, X_n$ have the joint probability density function

$$f(x) = \frac{(\det B)^{\frac{1}{2}}}{(2\pi)^{n/2}} e^{-\frac{1}{2}xBx'}$$

where $x = (x_1, x_2, \cdots, x_n)$ and B is a positive definite symmetric matrix. Suppose Q is a positive semi-definite symmetric matrix of rank $r \leq n$ and $L_1, L_2, \cdots, L_p$ is a set of symmetric matrices. We wish to obtain the joint distribution function $G(\xi_1, \xi_2, \cdots, \xi_p)$ of the p ratios

$$\frac{XL_1X'}{XQX'}, \quad \frac{XL_2X'}{XQX'}, \cdots, \frac{XL_pX'}{XQX'},$$

where $X = (X_1, X_2, \cdots, X_n)$.

The existence of such an orthogonal matrix S that $SQS' = I^{(r)}$, where $I^{(r)}$ is the diagonal matrix having the first r diagonal elements equal to unity and the rest equal to zero, is well-known. Let $X = YS$, $C = SBS'$, $M_i = SL_iS'$, and note that C and the M_i are symmetric matrices. Also

$$G(\xi_1, \xi_2, \cdots, \xi_p) = P\left\{\frac{YM_1Y'}{YI^{(r)}Y'} \leq \xi_1; \cdots; \frac{YM_pY'}{YI^{(r)}Y'} \leq \xi_p\right\},$$

where $Y = (Y_1, Y_2, \cdots, Y_n)$ has the probability density function

$$g(y) = \frac{(\det C)^{\frac{1}{2}}}{(2\pi)^{n/2}} e^{-\frac{1}{2}yCy'},$$

and $y = (y_1, y_2, \cdots, y_n)$.

Suppose the L_i mutually commute in pairs. Then so do the M_i; for $M_iM_j = SL_iS'SL_jS' = SL_iL_jS' = SL_jL_iS' = SL_jS'SL_iS' = M_jM_i$, since S is orthogonal. Hence, there is an orthogonal matrix U which simultaneously reduces each M to diagonal form; that is, $N = UMU'$ is a diagonal matrix (cf. Weyl [8, p. 25]).

Let $Y = ZU; D = UCU'$, so that

$$ZN_iZ' = \sum_{j=1}^{n} \mu_{ji} Z_j^2$$

$$G(\xi_1, \xi_2, \cdots, \xi_p) = P\left\{ \frac{\Sigma \mu_{ji} Z_j^2}{\Sigma \nu_j^{(r)} Z_j^2} \leq \xi_1 ; \cdots ; \frac{\Sigma \mu_{jp} Z_j^2}{\Sigma \nu_j^{(r)} Z_j^2} \leq \xi_p \right\},$$

where $\nu_j^{(r)} = 1$ if $j \leq r$;

$= 0$ if $j > r$,

and $Z = (Z_1, Z_2, \cdots, Z_n)$ has the probability density function

$$h(z) = \frac{(\det D)^{\frac{1}{2}}}{(2\pi)^{n/2}} e^{-\frac{1}{2}zDz'},$$

where $z = (z_1, z_2, \cdots, z_n)$.

We can now apply the results of section 6. If $\psi(t_1, t_2, \cdots, t_n)$ is the characteristic function corresponding to the joint distribution function of $Z_1^2, Z_2^2, \cdots, Z_n^2$ it is clear that

$$\psi(t_1, t_2, \cdots, t_n) = \left[\frac{\det D}{\det(D - 2iT)}\right]^{\frac{1}{2}},$$

where T is the diagonal matrix whose diagonal elements are $t_1, t_2, \cdots, t_n$. Applying (15), with $\phi = \psi$, we obtain, since G is obviously a continuous distribution function

$$(-1)^{p+1} 2^p G(\xi_1, \xi_2, \cdots, \xi_p)$$

$$= A_0 + \sum_{k=1}^{p} \frac{A_k}{(\pi i)^k} \sum_{i_1 < i_2 < \cdots < i_k} \oint \oint \cdots \oint \psi \left\{ \sum_{l=1}^{k} w_l(\mu_{1i_l} - \nu_1^{(r)} \xi_{i_l}), \cdots, \sum_{l=1}^{k} w_l(\mu_{ni_l} - \nu_n^{(r)} \xi_{i_l}) \right\} \frac{dw_1 \cdots dw_k}{w_1 w_2 \cdots w_k} \tag{17}$$

$$= A_0 + \sum_{k=1}^{p} \frac{A_k}{(\pi i)^k} \sum_{i_1 < i_2 < \cdots < i_k} \oint \oint \cdots \oint \left[\frac{\det D}{\det\left\{ d_{\alpha\beta} - 2i\delta_{\alpha\beta} \sum_{l=1}^{k} w_l(\mu_{\beta i_l} - \nu_\beta^{(r)} \xi_{i_l}) \right\}}\right]^{\frac{1}{2}} \frac{dw_1 dw_2 \cdots dw_k}{w_1 w_2 \cdots w_k},$$

where $D = [d_{\alpha\beta}]$ and $\delta_{\alpha\beta}$ is the Kronecker delta.

It is, of course, evident that a result analogous to (17) could be obtained, by considering p ratios

$$\frac{XL_1X'}{XQ_1X'}, \quad \frac{XL_2X'}{XQ_2X'}, \cdots, \frac{XL_pX'}{XQ_pX'},$$

where the $2p$ matrices $L_1, L_2, \cdots, L_p, Q_1, Q_2, \cdots, Q_p$ are symmetric and mutually commute in pairs, and $Q_1, Q_2, \cdots, Q_p$ are positive semi-definite.

In the case $p = 1$ in (17) and for special classes of matrices L_1, Q_1, B the calculus of residues may be employed to obtain closed expressions for the distribution of

$$\frac{XL_1X'}{XQ_1X'}.$$

Formula (17) can be applied to obtain the joint distribution of serial correlation coefficients with different lags. The author plans to incorporate these results with those mentioned at the end of section 4 in a forthcoming paper, written jointly with Roy B. Leipnik.

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