# BOUNDS FOR SOME FUNCTIONS USED IN SEQUENTIALLY TESTING THE MEAN OF A POISSON DISTRIBUTION[1]

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**1. Introduction.** Let $z = \log \frac{f(x, \lambda_1)}{f(x, \lambda_0)}$, where $f(x, \lambda_i) = (e^{-\lambda_i} \lambda_i^x)/x!$, $(i = 0, 1)$, is the elementary probability law of a Poisson variate $X$, under the hypothesis that the mean is equal to $\lambda_i$. Without loss of generality we shall assume $\lambda_1 > \lambda_0$.

Let $H_0$ be the hypothesis that the distribution of $X$ is given by $f(x, \lambda_0)$. Wald [1, pp. 286–287] has devised general upper and lower bounds for the probability of accepting $H_0$, when $\lambda$ is the true value of the parameter, and the sequential probability ratio test is used. This probability is called the operating-characteristic function and is designated by $L(\lambda)$. Using these results he has computed the bounds for the binomial and normal distributions [2, pp. 137–142]. We shall do the same thing for the Poisson distribution, since the restrictions [1, p. 284, conditions I to III] under which these general limits are valid can rather easily be shown to apply to the Poisson distribution, if we make the further restriction that $E(z) \neq 0$.

These general results are

$$\frac{1 - B^h}{\delta A^h - B^h} \leq 1 - L(\lambda) \leq \frac{1 - \eta B^h}{A^h - \eta B^h}, \qquad \text{if } h > 0,$$

and

$$\frac{1 - A^h}{\delta B^h - A^h} \leq L(\lambda) \leq \frac{1 - \eta A^h}{B^h - \eta A^h}, \qquad \text{if } h < 0, \tag{1}$$

where $\alpha$, $\beta$ are probabilities of committing errors of the first and second kind respectively and

$$\text{A} = (1 - \beta)/\alpha, \qquad \text{B} = \beta/(1 - \alpha)$$

$$\eta = \operatorname*{glb}_{\zeta} \zeta E\left(e^{hz} \mid e^{hz} < \frac{1}{\zeta}\right), \qquad \zeta > 1; \tag{2}$$

$$\delta = \operatorname*{lub}_{\rho} \rho E\left(e^{hz} \mid e^{hz} \geq \frac{1}{\rho}\right), \qquad 0 < \rho < 1;$$

and $h$ is the non-zero root of the expression, $Ee^{zt} = 1$. Hence the only remaining unknowns are $\eta$ and $\delta$.

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The following bounds to $En$, the expected number of observations required by the sequential probability ratio test defined by $\alpha$, $\beta$ have been derived [1, pp. 143–147]:

$$\frac{L(\lambda)(\log B + \xi') + [1 - L(\lambda)] \log A}{Ez} \underset{\geq}{\leq} En \underset{\geq}{\leq} \frac{L(\lambda) \log B + [1 - L(\lambda)](\log A + \xi)}{Ez},$$

the upper or lower inequality signs holding according as $Ez > 0$ or $Ez < 0$, where

$$\xi' = \operatorname*{Min}_{r} E(z + r \mid z + r \leq 0), \tag{3}$$

and

$$\xi = \operatorname*{Max}_{r} E(z - r \mid z - r \geq 0), \qquad (r \geq 0)\,. \tag{4}$$

Using the limits to $L(\lambda)$, we then find $\xi$ and $\xi'$, which determine $En$.

**2. Special terminology.** By an *almost-increasing* function we shall mean one that has the following properties: If $x$ is any point of discontinuity, then (a) $x + k$ is also where $k$ is any integer and $x + l$ is a point of continuity if $l$ is not integral, (b) $f(x - \epsilon) < f(x - \epsilon') < f(x)$ for $0 < \epsilon' < \epsilon < 1$, (c) $f(x - 1) < f(x)$, (d) $\lim_{\epsilon \to 0} f(x + \epsilon) = f(x +) < f(x)$, (e) $f(x - 1 +) < f(x +)$. It is clear that the minimum value for $f(y)$ in any closed interval $[a, b]$ is equal to $\min [f(a), f(a' +)]$ where $a'$ is defined as $a$ if the closed interval contains no discontinuity, and as the leftmost point of discontinuity otherwise. As special cases, if $a$ is a point of discontinuity this minimum is $f(a +)$ and if $x < a < b < x + 1$ the minimum is $f(a)$.

*Almost-decreasing* functions are defined similarly except that the inequalities go the other way. In this case the maximum in the interval is $\max[f(a), f(a' +)]$ and we have special cases as above.

**3. The case $h > 0$.** Since $e^z = a^x e^{-c}$, where $a = \lambda_1/\lambda_0$ and $c = (\lambda_1 - \lambda_0)$ the condition $e^{hz} \leq 1/\zeta$ may be expressed as $a^{hx} e^{-ch} \leq 1/\zeta$, whence

$$x \leq c/\log a - \log \zeta/(h \log a) = s - r \text{ (say)}. \tag{5}$$

Since $x \geq 0$, $r \leq s$. Hence $0 < r \leq s$. Also

$$Ee^{zh} = \sum_{x=0}^{\infty} (e^{-c} a^x)^h \frac{e^{-\lambda} \lambda^x}{x!} = \exp(-ch - \lambda + \lambda a^h), \tag{6}$$

and

$$\zeta E(e^{zh} \mid e^{zh} \leq 1/\zeta) = \zeta E[(e^{-c} a^x)^h \mid x \leq s - r]. \tag{7}$$

From (5), $\zeta = a^{rh}$ and (7) becomes

$$(7.1)\qquad a^{rh}\,\frac{\sum_{x=0}^{[s-r]} \frac{e^{-\lambda}\lambda^x}{x!} e^{-ch} a^{xh}}{\sum_{x=0}^{[s-r]} \frac{e^{-\lambda}\lambda^x}{x!}};$$

where $[s - r]$ is the largest integer $\leq (s - r)$. Our problem is to minimize (7) with respect to $\zeta$. Since $r$ is a strictly increasing function of $\zeta$, this is equivalent to minimizing $a^{rh}C/D = \theta$ (say) with respect to $r$, where

$$C = \sum_{x=0}^{[s-r]} \frac{\lambda^x a^{xh}}{x!}, \qquad \text{and} \qquad D = \sum_{x=0}^{[s-r]} \frac{\lambda^x}{x!}.$$

It will be shown that (7.1) is an almost-increasing function of $r$ and therefore the minimum occurs at either $r = 0$ or $r = \nu +$, where $\nu = s - [s]$, since the saltuses occur at $r = \nu + k$ for $k = 0, 1, 2, \cdots, [s]$.

Since $a^{rh}$ is an increasing function of $r$ and $C/D$ remains constant as long as $[s - r]$ remains constant, condition (b) is fulfilled.

Conditions (c) to (e) refer to the saltuses only, hence, to show them, we may assume, without loss of generality that $r$ and $s$ are integral. We proceed by induction, using the notation $\theta(w)$ to mean the value of $\theta$, when $r = w$, to show (c).

First we prove the following:

LEMMA A. $\theta(s) > \theta(s - 1)$.

PROOF: Since we assumed $\lambda_1 > \lambda_0$ and $h > 0$, $a^h > 1$. Hence $(1 + \lambda)a^h > 1 + \lambda a^h$, whence, *a fortiori*, $a^{sh} > a^{(s-1)h}(1 + \lambda a^h)/(1 + \lambda)$.

To show that if $\theta(r + 1) > \theta(r)$, then $\theta(r) > \theta(r - 1)$, we shall show that

$$(8)\qquad CD + Dba^{(n+1)h} < CDa^h + Cb$$

implies

$$(9)\qquad CD + Dbqa^{(n+1)h} < CDa^h + Cbqa^h,$$

where $n = s - r$, $b = \lambda^n/n!$, $q = \lambda/(n + 1)$.

Since, as we shall see below,

$$(10)\qquad Dba^{(n+1)h}(q - 1) < Cb(qa^h - 1),$$

or

$$(11)\qquad Da^{(n+1)h}(q - 1) < C(qa^h - 1),$$

addition of (8) and (10) yields the desired result, (9).

It now remains to prove (11) or that

$$(12)\qquad \left[\sum_{x=0}^{n} \frac{\lambda^x}{x!}\right] a^{(n+1)h} (\lambda - n - 1) < \left[\sum_{x=0}^{n} \frac{\lambda^x a^{xh}}{x!}\right] (\lambda a^h - n - 1).$$

Setting (6) equal to 1 we get $\lambda a^h = ch + \lambda$, which when substituted in (12) yields

$$(ch + \lambda)^{n+1}(\lambda - n - 1) \sum_{x=0}^{n} \frac{\lambda^x}{x!} < \lambda^{n+1}(ch + \lambda - n - 1) \sum_{x=0}^{n} \frac{(ch + \lambda)^x}{x!}.$$

Upon letting $p = ch + \lambda$, we have

$$\frac{\lambda - (n + 1)}{\lambda^{n+1}} \sum_{x=0}^{n} \frac{\lambda^x}{x!} < \frac{p - (n + 1)}{p^{n+1}} \sum_{x=0}^{n} \frac{p^x}{x!} = F(p), \qquad \text{say.}$$

Then our problem reduces to showing that $F(y)$ is increasing in $0 < \lambda \le y \le p$ or that the derivative with respect to $y$, $F'(y)$ is positive.

$$\begin{aligned} F'(y) = - \sum_{x=0}^{n-1} \frac{(n - x)(y^{x-n-1})}{x!} &+ (n + 1) \sum_{x=0}^{n-1} \frac{(n - x)(y^{x-n-1})}{(x + 1)!} + (n + 1)^2 y^{-n-2} \\ &> (n + 1)^2 y^{-n-2}, \qquad \text{since } (n + 1) > (x + 1); \\ &> 0 \qquad \text{since } y > 0. \end{aligned}$$

Thus condition (c) is demonstrated. To show (d) we must show that $\theta(r +) < \theta(r)$, which means that

$$a^{rh} \frac{C - ba^{nh}}{D - b} < a^{rh} \frac{C}{D}.$$

But this is true if $C < Da^{nh}$ which is easily verfied. Condition (e) is equivalent to showing that

$$a^{(r-1)h} \frac{C}{D} < a^{rh} \frac{C - ba^{nh}}{D - b},$$

which is proved just as (c) was.

Hence,

$$\eta = \min \left\{ e^{-ch} \sum_{x=0}^{[s]} \frac{e^{-\lambda} \lambda^x a^{hx}}{x!} \Big/ \sum_{x=0}^{[s]} \frac{e^{-\lambda} \lambda^x}{x!}, \right. \quad (13)$$
$$\left. a^{\nu h} e^{-ch} \sum_{x=0}^{[s-1]} \frac{e^{-\lambda} \lambda^x a^{hx}}{x!} \Big/ \sum_{x=0}^{[s-1]} \frac{e^{-\lambda} \lambda^x}{x!} \right\}.$$

As special cases we have (i) if $s$ is integral, $\eta$ is the latter with $\nu = 0$ and (ii) if $s < 1$ (b) is the only applicable condition and we have an ordinary increasing function, hence $\eta$ is the former.

Similarly, it may be shown that

$$\delta = \max [e^{-ch} E(a^{xh} | x \ge \{s\}), \quad a^{-\mu h} e^{-ch} E(a^{xh} | x \ge \{s + 1\})], \quad (14)$$

where $\{s\}$ is the smallest integer $\ge s$ and $\mu = \{s\} - s$. Here there is only one special case, namely (i). If $h < 0$, $\delta$ is the larger of the two expressions on the right side of (13) and $\eta$ is the smaller of the two corresponding expressions in (14).

**4.** Since $z = -c + x \log a$, $\xi$ may be written

$$\operatorname*{Max}_t \log aE(x - t \mid x \geq t),$$

where $t = (r + c)/(\log a)$. Hence $s = c/\log a \leq t < \infty$. Therefore if we can show that $E(x - t \mid x \geq t) = \gamma(t)$ (say), is an almost-decreasing function of $t$ we will know that $\xi$ occurs either when $t = s$ or $\{s\} +$ since, as will be seen, the jumps occur at integral $t$.

To show (c) we make use of the following which is easily proven:

LEMMA B. *Let $X$, $Y$, $Z$ each be greater than zero. Then a necessary and sufficient condition that* $\dfrac{X}{Y} < \dfrac{X + Y}{Y + Z}$ *is that* $XZ < Y^2$.

Therefore, to show for integral $t$ that

$$\gamma(t) < \gamma(t - 1), \tag{15}$$

or that

$$\frac{\sum_{x=t}^{\infty} (x - t)\dfrac{\lambda^x}{x!}}{\sum_{x=t}^{\infty} \dfrac{\lambda^x}{x!}} < \frac{\sum_{x=t}^{\infty} (x - t)\dfrac{\lambda^x}{x!} + \sum_{x=t}^{\infty} \dfrac{\lambda^x}{x!}}{\sum_{x=t}^{\infty} \dfrac{\lambda^x}{x!} + \dfrac{\lambda^{t-1}}{(t - 1)!}},$$

we need only show that, for all integral $t$,

$$\frac{\lambda^{t-1}}{(t - 1)!} \sum_{x=t}^{\infty} \frac{(x - t)\lambda^x}{x!} < \left[\sum_{x=t}^{\infty} \frac{\lambda^x}{x!}\right]^2. \tag{16}$$

Since both sides of (16) are power series in $\lambda$ where the exponents start with $2t$ we need only show that the coefficient of every term on the left is less than the corresponding term on the right.

In the case of the coefficient of $\lambda^{2j+2t}$, $(j \geq 0)$ we have to show that

$$\frac{2j + 1}{(t + 2j + 1)!(t - 1)!} < \frac{2}{(t + 2j)!t!} + \frac{2}{(t + 2j - 1)!(t + 1)!} + \cdots + \frac{1}{(t + j)!(t + j)!},$$

or by multiplying both sides by $(2t + 2j)!$ that

$$(2j + 1)\binom{2t + 2j}{t - 1} < 2\binom{2t + 2j}{t} + 2\binom{2t + 2j}{t + 1} + \cdots + 2\binom{2t + 2j}{t + j - 1} + \binom{2t + 2j}{t + j} = \text{M, say.}$$

Replacing all the binomial coefficients on the right by the smallest one we have

$$(2j + 1)\binom{2t + 2j}{t - 1} < (2j + 1)\binom{2t + 2j}{t} < M,$$

since $\binom{n}{s-1} < \binom{n}{s}$ for $n \geq 2s$. Thus the truth of (16) has been established for even exponents. The odd terms are treated similarly.

Hence, we have shown that $\gamma(t)$ is a strictly decreasing function of $t$, if $t$ takes on integral values only. We shall now show (b), i.e. that

$$\gamma(t) = \frac{\sum_{x=t}^{\infty} (x-t)\frac{\lambda^x}{x!}}{\sum_{x=t}^{\infty} \frac{\lambda^x}{x!}} < \frac{\sum_{x=\{t-\epsilon\}}^{\infty} (x-t+\epsilon)\frac{\lambda^x}{x!}}{\sum_{x=\{t-\epsilon\}}^{\infty} \frac{\lambda^x}{x!}} = \gamma(t-\epsilon). \tag{17}$$

The denominators are equal and each term of the numerator on the right is greater than the corresponding term on the left, hence (17) is valid.

Conditions (a) and (d) can be shown, by showing in a similar manner, that

$$\gamma(t+) = 1 + \gamma(t+1) \tag{18}$$

and $\gamma(t) > 1 + \gamma(t+1)$ for integral $t$. By using (18) for $t$ and $t - 1$ together with (15) we show $\gamma(t - 1 +) < \gamma(t +)$, which is condition (e). Thus we have shown that

$$\xi = \max \begin{cases} -c + \log a \sum_{x=\{s\}}^{\infty} \frac{x\lambda^x e^{-\lambda}}{x!} \Big/ \sum_{x=\{s\}}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!}, \\ \log a \left[ -\{s\} + \sum_{x=\{s+1\}}^{\infty} \frac{x\lambda^x e^{-\lambda}}{x!} \Big/ \sum_{x=\{s+1\}}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} \right]. \end{cases}$$

As in Section 3, $\xi'$ is the lower analogue of $\xi$, i.e.

$$\xi' = \min \{-c + E(x \mid x \leq [s]), -[s]\log a + E(x \mid x \leq [s-1])\},$$

and the special cases are as in that section.

## REFERENCES


[1] A. WALD, "On cumulative sums of random variables," *Annals of Math. Stat.*, Vol. 15 (1944), pp. 283–296.

[2] A. WALD, "Sequential tests of statistical hypotheses," *Annals of Math. Stat.*, Vol. 16 (1945), pp. 117–186.