ON THE LIMITING DISTRIBUTIONS OF ESTIMATES BASED ON SAMPLES FROM FINITE UNIVERSES[1]

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1. Summary. The paper shows that under very broad conditions the usual theorems concerning the limiting distributions of estimates hold for estimates based on samples selected from finite universes, at random without replacement. It may be remarked that under the same conditions, the same conclusions are true for random sampling from finite universes with replacement, if the universes are permitted to change within the limitations set by condition W.

2. Introduction. It has long been known that the limiting distribution of arithmetic means of samples selected at random with replacement from finite universes, or from infinite universes is normal under very general conditions. When, however, a sample is selected from a finite universe without replacement, and the size of the sample as compared with that of the universe is too large for the universe to be treated as infinite, the proof that the limiting distribution of the mean is normal appears to have been given only for the case where the universe is multinomial.[2] In this paper we prove that the limiting distribution of the mean is normal provided only that as the universe increases in size, the higher moments do not increase too rapidly as compared with the variance, and that for sufficiently large sizes of sample and population the ratio of size of sample to size of universe is bounded away from 1. Various extensions are given, but these are almost immediate consequences of the theorem on the limiting distribution of the mean.

The method used is that of showing that the moments of the standardized mean tend to those of the normal distribution. In doing this we generalize a theorem of Wald and Wolfowitz,[3] by making it applicable to permutations of samples from finite populations, and by reducing a little the conditions on the coefficients. The theorem on the mean is then a simple corollary.

We also note that with these proofs on limiting distributions we can make the corresponding assertions concerning characteristic functions. Although no applications of this fact are given, it seems likely that some useful results could be obtained.

3. Preliminary lemmas. In calculating the k-th moments and their limits we

[1] Presented to the American Mathematical Society at a meeting held in New York City on April 17, 1948.

[2] See F. N. David, "Limiting distributions connected with certain methods of sampling human populations," *Stat. Res. Mem.*, Vol. 2 (1938), pp. 69–90, especially p. 77.

[3] A. Wald and J. Wolfowitz, "Statistical tests based on permutations of the observations," *Annals of Math. Stat.*, Vol. 5 (1944), pp. 358–372, especially p. 359.

shall use an infrequently given form of the multinomial expansion and some properties of symmetric polynomials. In this section we make the necessary definitions, and present four lemmas embodying the results we shall use.[4]

A t-partition of a positive integer k consists of t positive integers $\alpha_1, \cdots, \alpha_t$ such that $\alpha_1 + \cdots + \alpha_t = k$. Two t-partitions $\alpha_1, \cdots, \alpha_t$ and $\beta_1, \cdots, \beta_t$ of k will be said to be distinct if for at least one value of h we have $\alpha_h \neq \beta_h$.

Let $\varphi(\alpha_1, \cdots, \alpha_t)$, written $\varphi(\alpha)$, be any function of the t-partitions of k. By $\Sigma_{1t}\varphi(\alpha)$ we shall mean the summation of $\varphi(\alpha_1, \cdots, \alpha_t)$ over all distinct t-partitions of k.

By $\Sigma_{2t}\varphi(\alpha)$ we shall mean the summation of $\varphi(\alpha)$ over all distinct permutations of $\alpha_1, \cdots, \alpha_t$.

By $\Sigma_{3t}\varphi(\alpha)$ we shall mean the summation of $\varphi(\alpha)$ over all distinct t partitions of k satisfying the condition $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_t$.

Let $\psi(\nu_1, \cdots, \nu_t)$ be any function of the variables $\nu_1, \cdots, \nu_t$. Then by $\Sigma_{4n}\psi(\nu_1, \cdots, \nu_t)$ we shall mean the summation of $\psi(\nu_1, \cdots, \nu_t)$ over all possible selections of t integers from 1 to n arranged so that $\nu_1 > \nu_2 > \cdots > \nu_t$.

The formula for the multinomial given below is not presented as a new result. It is given only as a means of referring to the result we need.

LEMMA 1. *Let $\xi_1, \cdots, \xi_n$ be any quantities or random variables and let k be a positive integer. Then*

$$(\xi_1 + \cdots + \xi_n)^k = \sum_{t=1}^{k} \Sigma_{1t}\, C^k_{\alpha_1\cdots\alpha_t}\, \Sigma_{4n}\, \xi_{\nu_1}^{\alpha_1} \cdots \xi_{\nu_t}^{\alpha_t},$$

where

$$C^k_{\alpha_1\cdots\alpha_t} = \frac{k!}{\alpha_1! \cdots \alpha_t!}.$$

The proof is omitted.

The following lemma will be useful in connection with several of the results of this section:

LEMMA 2. *If $\varphi(\alpha)$ is a function of the t-partitions of k, then*

$$\Sigma_{1t}\varphi(\alpha) = \Sigma_{3t}\Sigma_{2t}\varphi(\alpha).$$

The verbalization of the lemma is practically its proof.

Let us now define certain symmetric polynomials that we shall use.

Let $S_{\alpha_1,\cdots,\alpha_t} = \Sigma\xi_{\nu_1}^{\alpha_1} \cdots \xi_{\nu_t}^{\alpha_t}$ where the α's are positive integers and the summation extends over all possible arrangements $\nu_1, \cdots, \nu_t$ of t of the integers $1, \cdots, N$. Hence there will be $N^{(t)} = N(N-1)\cdots(N-t+1)$ terms in $S_{\alpha_1,\cdots,\alpha_t}$.

LEMMA 3. *Suppose that $t_1, \cdots, t_h$ are an h partition of t, that*

$$\alpha_{t_1+\cdots+t_{i-1}+1} = \cdots = \alpha_{t_1+\cdots+t_i}, \quad (i = 1, \cdots, h;\, t_0 = 0),$$

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[4] The order of sections 3 and 4 is largely a matter of taste; some may prefer to treat section 3 as an appendix to section 4 to be referred to when necessary.

and that

$$\alpha_1 \neq \alpha_{t_1+1} \neq \cdots \neq \alpha_{t_1+\cdots+t_{h-1}+1} .$$

Then, defining

$$(3.1) \qquad S'_{\alpha_1\cdots\alpha_t} = \Sigma_{2t}\Sigma_{4n}\xi_{\nu_1}^{\alpha_1} \cdots \xi_{\nu_t}^{\alpha_t} ,$$

it follows that

$$S_{\alpha_1,\cdots,\alpha_t} = t_1! \cdots t_h! S'_{\alpha_1,\cdots,\alpha_t} .$$

To prove Lemma 3, it is only necessary to note that each term of $S'_{\alpha_1,\cdots,\alpha_t}$ will determine $t_1! \cdots t_h!$ equal terms of $S_{\alpha_1,\cdots,\alpha_t}$.

Although the moments that we shall obtain will be functions of $S_{\alpha_1,\cdots,\alpha_t}$, the condition that we shall use on the moments can be interpreted directly only in terms of S_j . Consequently, in order to be able to analyze the implications of that condition on $S_{\alpha_1,\cdots,\alpha_t}$, we state the following lemma:

LEMMA 4. *The symmetric polynomial $S_{\alpha_1,\cdots,\alpha_t}$ is equal to a sum of products of the form*

$$\pm S_{\gamma_1} S_{\gamma_2} \cdots S_{\gamma_h}$$

where $\gamma_1 , \cdots , \gamma_h$ are an h-partition of k, $h \leq t$, and each γ is a sum of one or more of the α's. Furthermore, if $S_1 = 0$, then $h \leq [k/2]$ where $[k/2] = k/2$ if k is even and $[k/2] = (k - 1)/2$ if k is odd. This follows from the result

$$(3.2) \quad S_{\alpha_t} \cdot S_{\alpha_1,\cdots,\alpha_{t-1}} = S_{\alpha_1,\cdots,\alpha_t} + S_{\alpha_1+\alpha_t,\alpha_2,\cdots,\alpha_{t-1}} + \cdots + S_{\alpha_1,\cdots,\alpha_{t-2},\alpha_{t-1}+\alpha_t} .$$

PROOF: It is easy to prove (3.2) by comparing terms. Then the other assertions follow from the repeated use of (3.2) and the resulting fact that each γ is a sum of one or more of the α's.

4. The limiting distribution. In this section we obtain the generalization of the theorem of Wald and Wolfowitz to which reference was made above.

Let $U_1 , U_2 , \cdots , U_N , \cdots$ be a sequence of universes, the universe U_N containing the elements[5] $x_{\nu N}$ and let the arithmetic mean of the elements of U_N be denoted by $\bar{x}_N$. Furthermore, let

$$\mu_{rN} = \mu_r(U_N) = \left(\frac{1}{N}\right) \sum_\nu (x_{\nu N} - \bar{x}_N)^r.$$

Let $C_1 , C_2 , \cdots , C_n , \cdots$ be a sequence of sets of coefficients, the set C_n containing the elements c_{jn} and let the arithmetic mean of the elements of C_n be denoted by $\bar{c}_n$. We exclude the possibility that the elements of any C_n all vanish, and hence we can suppose that $\sum_j c_{nj}^2 = 1$. Furthermore, let

[5] The letter ν will assume all integral values from 1 to N. The letter r will assume all positive integral values. The letter j will assume all integral values from 1 to n. The letter t will assume all integral values from 1 to k. The symbol lim will stand for the limit as n or N or both, as the case may be, increase without limit, it being understood that lim $n/N < 1$.

$$\mu'_{rn} = \mu'_r(C_n) = \left(\frac{1}{n}\right) \sum_j c_{jn}^r .$$

Since $\sum_j (c_{jn} - \bar{c}_n)^2 > 0$, it follows that, if we define $A_n = n^{1/2}\bar{c}_n$, then $A_n^2 \leq 1$.

Let n elements be selected at random without replacement from U_N and let us denote these elements by x'_{jn}, the subscript j indicating the order of selection, i.e., x'_{in} is the i-th element of U_N selected for the sample even though it may be x_{NN}.

The linear function that we shall study is

$$z_n = c_{1n}x'_{1N} + \cdots + c_{nn}x'_{nN},$$

i.e., the value of z_n is determined by multiplying the j-th element selected for the sample by c_{jn} and summing for j. Then, since $Ex'_{in} = \bar{x}_N$, we have

$$Ez_n = n\bar{x}_N\bar{c}_n .$$

Furthermore,

$$\sigma_{z_n}^2 = \left(\frac{N}{N-1}\right) \mu_{2N} \left(1 - \frac{n}{N} A_n^2\right).$$

To see this we first note that

$$\sum_{i \neq j = 1}^{n} c_{in}\, c_{jn} = n^2 \bar{c}_n^2 - 1,$$

$$E(x'_{in} - \bar{x}_N)^2 = \mu_{2N},$$

and, if $i \neq j$,

$$E(x'_{in} - \bar{x}_N)(x'_{jn} - \bar{x}_N) = -\mu_{2N}\left(\frac{1}{N-1}\right).$$

From the definition of variance we have

$$\sigma_{z_n}^2 = E(z_n - Ez_n)^2 = \sum_{i,j=1}^{n} c_{in}\, c_{jn}\, E(x'_{in} - \bar{x}_N)(x'_{jn} - \bar{x}_N),$$

and making the indicated substitutions the result follows from a few simple manipulations.

If we define $\tilde{x}_n$ to be the arithmetic mean of $x'_{1N}, \cdots, x'_{nN}$, then it follows that $\sqrt{n}\, c_{jn} = 1$ and, as is well known,

$$E\tilde{x}_n = \bar{x}$$

$$\sigma_{\tilde{x}_n}^2 = \left(\frac{N-n}{N-1}\right) \frac{\mu_{2n}}{n}.$$

Hence, if we can find the limiting distribution of

$$Z_n = \frac{z_n - Ez_n}{\sigma_{z_n}},$$

then the limiting distribution of $(\tilde{x} - \bar{x})/\sigma_{\tilde{x}}$ will be a special case.

We shall need to place some sort of limitation on the sequences U_N and C_n if we are to obtain theorems on limiting distributions of statistics based on them.

The condition W that we shall use is satisfied by a slightly larger class of sequences U_N and C_n than that of Wald and Wolfowitz because it does not rule out the possibility that all the elements of C_n should be equal. It should be noted, however, that for their purposes this extension of the class of sequences satisfying U_N and C_n is vacuous since they required $n = N$, so that in their case if all the elements of C_n were equal, say k/N, we would have $z_N = k\,\bar{x}_N$ no matter in what order the elements of U_N were selected for the sample.

CONDITION W. The sequence U_N and C_n will satisfy the condition W if

$$\mu_{rN} = \mu_{2N}^{r/2}\,\lambda_r(N),$$

$$\mu'_{rn} = n^{-r/2}\,\lambda'_r(n),$$

and

$$\frac{nA_n^2}{N} < 1 - \epsilon,$$

for sufficiently large n and N, where a finite value λ exists such that for all r

$$\sup \mid \lambda_r(N) \mid < \lambda,$$

$$\sup \mid \lambda'_r(n) \mid < \lambda,$$

and $\epsilon > 0$.

(Note that if W is satisfied for all even values of r then W is also satisfied for all odd values of r since $\mu_{r+2}\mu_r \geq \mu_{r+1}^2$).

A general theorem on moments is the following:

THEOREM 1. *Let $S_{\alpha_1,\cdots,\alpha_t}$ and $S'_{\alpha_1,\cdots\alpha_t}$ be defined in terms of $x_N - \bar{x}_N$ instead of ξ_ν and let $T'_{\alpha_1,\cdots,\alpha_t}$ be the same function of the c_{jn} that $S'_{\alpha_1,\cdots,\alpha_t}$ is of the $x_{\nu N} - \bar{x}_N$. Furthermore, let $E_k = EZ_n^k$. Then*

$$E_k = \sum_t \sum\nolimits_{3t} C^k_{\alpha_1\cdots\alpha_t} \frac{S_{\alpha_1\cdots\alpha_t}\,T'_{\alpha_1\cdots\alpha_t}}{N^{(t)}\,\sigma^k_{z_n}}. \tag{4.1}$$

PROOF: From the definition of Z_n and Lemma 1, it follows that

$$\sigma^k_{z_n} E_k = \sum_t \sum\nolimits_{1t} C^k_{\alpha_1\cdots\alpha_t} \sum\nolimits_{4n} c^{\alpha_1}_{\nu_1 n}\cdots c^{\alpha_t}_{\nu_t n}\, E(x'_{\nu_1 N} - \bar{x}_N)^{\alpha_1}\cdots(x'_{\nu_t N} - \bar{x}_N)^{\alpha_t}.$$

Since we are selecting at random without replacement it follows that

$$N^{(t)} E(x'_{\nu_1 N} - \bar{x}_N)^{\alpha_1} \cdots (x'_{\nu_t N} - \bar{x}_N)^{\alpha_t} = S_{\alpha_1\cdots\alpha_t}.$$

If we now use Lemma 2 to replace Σ_{1t} by $\Sigma_{3t}\Sigma_{2t}$, we then obtain

$$\sigma^k_{z_n} N^{(t)}\, E_k = \sum_t \sum\nolimits_{3t} C^k_{\alpha_1\cdots\alpha_t} S_{\alpha_1\cdots\alpha_t} \sum\nolimits_{2t} \sum\nolimits_{4n} c^{\alpha_1}_{\nu_1 n} \cdots c^{\alpha_t}_{\nu_t n},$$

since both $C^k_{\alpha_1,\cdots,\alpha_t}$ and $S_{\alpha_1,\cdots,\alpha_t}$ are invariant under permutations of $\alpha_1, \cdots, \alpha_t$. Then from (3.1) and the definition of $T'_{\alpha_1,\cdots,\alpha_t}$, it follows that (4.1) is proved.

Our fundamental theorem is:

THEOREM 2. *If the sequences U_N and C_n satisfy the condition W, then*

$$\lim E_{2j+1} = 0,$$

and

$$\lim E_{2j} = \frac{(2j)!}{2^j \cdot j!},$$

so that, for any a,

$$\lim P\{Z_n < a\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{(-x^2/2)}\, dx,$$

PROOF: We wish to show that $\lim E_k$ exists and has the values given above First consider the parts of the typical term of E_k that depend on n and N, i.e., the expression

$$B = \frac{S_{\alpha_1, \cdots, \alpha_t} T'_{\alpha_1, \cdots, \alpha_t}}{N^{(t)} \mu_{2N}^{k/2} (N/N - 1)^{k/2} (1 - nA_n^2/N)^{k/2}}.$$

Since $\lim E_k$ will be the sum of the limits of a finite number of these terms, let us first determine under what conditions B will tend to zero as n and N become infinite.

From Lemma 4 it follows that

$$S_{\alpha_1, \cdots, \alpha_t} = \Sigma \pm S_{\gamma_1} S_{\gamma_2} \cdots S_{\gamma_h},$$

where $\gamma_1 + \cdots + \gamma_h = \alpha_1 + \cdots + \alpha_t$ and each of the γ's is the sum of one or more of the α's. From the definition of $S_{\alpha_1, \cdots, \alpha_t}$ in terms of $x_{\nu N} - \bar{x}_N$ it follows that $S_1 = 0$. Hence the minimum value of all γ's in any non-vanishing term of the summation is 2. Consequently we can say that for all non-vanishing terms $h \leq [k/2]$ and $h \leq t$. Finally if condition W is satisfied then

$$S_{\gamma_1} \cdots S_{\gamma_h} = N^h \mu_{2N}^{k/2} \lambda_h(N)$$

where

$$\sup | \lambda_h(N) | < \lambda^h.$$

Similarly

$$T'_{\alpha_1, \cdots, \alpha_t} = \Sigma \pm T_{\gamma_1}, \cdots, T_{\gamma_g},$$

where it may be that $T_1 \neq 0$ so that we cannot require $g \leq [k/2]$ for the term $T_{\gamma_1} \cdots T_{\gamma_g}$ to be non-vanishing. We still have, however, from Lemma 4 that $g \leq t$.

If condition W is satisfied, then

$$T_{\gamma_1} \cdots T_{\gamma_g} = n^{g-k/2} \lambda'_g(n),$$

where

$$\sup | \lambda_g'(n) | < \lambda^g.$$

Hence, from Lemma 4, the definitions of μ_{jN} and μ_{jn}' and condition W it follows that B is a sum (the number of terms does not depend on n or N) of terms like

$$D = \frac{N^h n^{g-k/2} \bar{\lambda}(N)\bar{\lambda}'(n)}{N^{(t)}(N/N - 1)^{k/2}(1 - nA_n^2/N)^{k/2}},$$

where

$$h \leq [k/2], \qquad h \leq t, \qquad g \leq t,$$

and

$$\sup | \bar{\lambda}(N) | < \infty,$$

$$\text{and } \sup | \bar{\lambda}'(n) | < \infty.$$

Since $h \leq t$, it follows that if $g < k/2$ then $\lim D = 0$. Hence, a possibly non-vanishing term must have $g \geq k/2$ and hence $t \geq k/2$ because $t \geq g$. Furthermore, $t \geq g + h - k/2$, since $h - k/2 \leq 0$ and $t \geqq g$. Hence $t - h \geq g - k/2$. Now, we can write

$$D = \frac{n^{g-k/2}}{N^{t-h}} \bar{\lambda}(N, n),$$

where

$$\sup | \bar{\lambda}(N, n) | < \infty,$$

since $nA_n^2/N < 1 - \epsilon$ for sufficiently large n and N.
Hence

$$\lim D = 0,$$

unless, perhaps, when $g - k/2 = t - h$, i.e., $h - k/2 = t - g$. Since $h - k/2 \leq 0$ and $t - g \geq 0$, *it follows that we must have* $h = k/2$ *and* $t = g$ *for* $\lim D$ *to be possibly not zero.*

If k is odd, then $h \leq (k - 1)/2$ and hence

$$\lim E_{2j+1} = 0,$$

since all terms obtained by expanding it as above will tend to zero.

If k is even, say $k = 2j$, and $\lim D$ is possibly non-vanishing, then h must equal j and we must have $\gamma_1 = \cdots = \gamma_j = 2$. Consequently, from Lemma 4, the only possibly non-vanishing terms of E_{2j} are those arising from the polynomials $S_{\alpha_1,\cdots,\alpha_t}$, $T'_{\alpha_1,\cdots,\alpha_t}$ with $\alpha_1 = \cdots = \alpha_s = 2$, and $\alpha_{s+1} = \cdots = \alpha_t = 1$, so that $2s + t - s = 2j$ or $t = 2j - s$, $s = 0, 1, \cdots, j$. For such values of $\alpha_1, \cdots, \alpha_t$ we have

$$C^k_{\alpha_1,\cdots,\alpha_t} = \frac{(2j)!}{2^s}.$$

Furthermore, as shown below, in developing $S_{\alpha_1,\cdots,\alpha_t}$ by means of Lemma 4 the coefficient of S_2^j is

$$(4.2) \qquad (-1)^{j-s}\frac{(2j-2s)!}{2^{j-s}(j-s)}.$$

Demonstration of (4.2): If $s = j$, then it follows from Lemma 4 that the coefficient of S_2^j is 1. If $s < j$, we use Lemma 4, and noting that $S_1 = 0$, we obtain

$$(4.3) \qquad S_{\alpha_1,\cdots,\alpha_t} = -S_{\alpha_1+\alpha_t,\alpha_2,\cdots,\alpha_{t-1}} - \cdots - S_{\alpha_1,\cdots,\alpha_{t-2},\alpha_{t-1}+\alpha_t},$$

where, since $\alpha_t = 1$, we have $\alpha_1 + \alpha_t = \alpha_2 + \alpha_t = \cdots = 1$, $\alpha_s + \alpha_t = 3$, and $\alpha_{s+1} + \alpha_t = \cdots = \alpha_{t-1} + \alpha_t = 2$. Consequently of the $t - 1$ terms of the above evaluation of $S_{\alpha_1,\cdots,\alpha_t}$, exactly s will have α's > 2 and $t - s - 1$ will be of the same form as $S_{\alpha_1,\cdots,\alpha_t}$ except that instead of s of the α's being 2 we have $s + 1$ of the α's equal 2. For each such s we repeat the process obtaining

$$S_{\alpha_1,\cdots,\alpha_t} = (-1)^{(t-s)/2}(t-s-1)(t-s-3)\cdots 3\cdot 1 \quad S_{\underbrace{2,\cdots,2}_{j}}$$

$$+ \text{ terms which have } h < j.$$

Consequently (4.2) provides the coefficient of S_2^j in $S_{\alpha_1,\cdots,\alpha_t}$. Since the other terms of $S_{\alpha_1,\cdots,\alpha_t}$ have $h < j$, they lead to terms of E_{2j} that vanish in the limit.

Furthermore, by Lemma 3, $T_{\alpha_1,\cdots,\alpha_t} = T'_{\alpha_1,\cdots,\alpha_t}s!(t-s)!$ and the only term of $T_{\alpha_1,\cdots,\alpha_t}$ for which $g = t$ is

$$T_2^s T_1^{t-s} = n^{(t-s)/2}A_n^{t-s}.$$

The other terms of $T_{\alpha_1,\cdots,\alpha_t}$ will lead to terms of E_{2j} that vanish in the limit since $g < t$. Consequently, eliminating terms known to tend to zero as n and N become infinite, we see that $E_{2j} - f(n, N)$ tends to zero as n and N become infinite, where

$$f(n, N) = \sum_{s=0}^{j}\frac{(2j)!}{2^s}(-1)^{j-s}\frac{(2j-2s)!\,N^j n^{j-s}A_n^{2j-2s}}{2^{j-s}s!(2j-2s)!\,N^{(2j-s)}(1-nA_n^2/N)^j}.$$

Now as n and N become infinite with $n < N$, we see that

$$\lim f(n, N) = \lim\frac{(2j)!}{2^j}\sum_{s=0}^{j}(-1)^{j-s}\frac{1}{s!(j-s)!}(nA_n^2/N)^{j-s}/(1-nA_n^2/N)^j$$

$$= \frac{(2j)!}{2^j j!},$$

i.e.,

$$\lim E_{2j} = \frac{(2j)!}{2^j\cdot j!}.$$

To complete the proof it is only necessary to note that the normal distribution is completely determined by its moments.[6]

[6] See for example, M. G. Kendall, *The Advanced Theory of Statistics*, Vol. I, London, Charles Griffin and Company, page 110.

Since Theorem 2 is a generalization of the Theorem of Wald and Wolfowitz, it is possible to generalize slightly all the applications they make of their theorem. The statements of these generalizations are omitted.

The application of Theorem 2 that led to this paper is the following: Suppose that $c_{jn} = n^{-1/2}$. Then the sequence C_n satisfies W and $A_n = 1$. Consequently we have proved

COROLLARY 1. *If the sequence U_N satisfies the condition W and if $\bar{x}_n$ is the arithmetic mean of a sample of n elements selected at random without replacement from U_N, then, for all a,*

$$\lim P\left\{\frac{n^{1/2}(\bar{x}_n - \bar{x}_N)}{\mu_{2N}^{1/2}(1 - m/n)^{\frac{1}{2}}} < a\right\} = \left(\frac{1}{2\pi}\right)\int_{-\infty}^{a} e^{-x^2/2}\, dx,$$

provided that $\epsilon > 0$ exists such that $n/N < 1 - \epsilon$, if n and N are sufficiently large.

Now the sequence of U_N will certainly satisfy W if U_N has the same moments for all values of N, or if the moments of U_N tend to fixed values as N increases, or if the universe U_N is a random sample of a universe having these properties. Consequently Theorem 1 and its corollaries will be valid for many applications, among them being the case studied by F. N. David[7] when U_N has the same multinomial distribution for each value of N.

The condition W is immediately satisfied for large classes of changing universes. For example, if the elements of all U_N are uniformly bounded and

$$\lim \mu_{2N} \neq 0,$$

then the condition W is satisfied. As an illustration, consider the case where U_N contains Np_N elements having the value one and $N(1 - p_N)$ elements having the value zero. Then

$$\mu_{2N} = p_N(1 - p_N),$$

and

$$\begin{aligned}\mu_{rN} &= \frac{1}{N}\sum_{r=1}^{Np_N}(1 - p_N)^r + \sum_{r=Np_N+1}^{N}(-p_N)^r,\\ &= p_N(1 - p_N)^r + (-1)^r(1 - p_N)p_N^r.\end{aligned}$$

Hence

$$\frac{\mu_{rN}}{\mu_{2N}^{r/2}} = \frac{(1 - p_N)^{r/2}}{p_N^{r/2-1}} + (-1)^r\frac{p_N^{r/2}}{(1 - p_N)^{r/2-1}},$$

so that condition W will be satisfied if $\epsilon > 0$ exists such that $\epsilon < p_N < 1 - \epsilon$ for all sufficiently large N.

Hence the limiting distribution of Z_n will be normal no matter how the proportions p_N change provided only that the universe U_N does not come to consist essentially only of zeros or only of ones.

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[7] Op. cit.

Various multivariate extensions of Theorem 2 are immediate. For example:

THEOREM 3. *Suppose that the elements of U_N are vectors of two components,*[8] $(x_{\nu N1}, x_{\nu N2})$, *and that the condition W is satisfied by the sequences C_n, U_{N1}, and U_{N2} where U_{Nh}, $h = 1, 2$, contains the elements $x_{\nu Nh}$.*

Let

$$z_{nh} = \sum_j c_{jn} x'_{jnh},$$

and let

$$Z_{nh} = \frac{z_{nh} - Ez_{nh}}{z_{nh}},$$

where the random variables x'_{jnh} are defined as were x'_{jn}.

Let

$$\rho_N = \frac{\Sigma_\nu (x_{\nu N1} - \bar{x}_{N1})(x_{\nu N2} - \bar{x}_{N2})}{(\mu_{2N1} \cdot \mu_{2N2})^{1/2}},$$

and suppose that lim ρ_N *exists and is equal to ρ where $\rho > -1 + \epsilon$. Then, the limiting distribution of Z_{n1} and Z_{n2} is bivariate normal with means 0, variances 1, and correlation coefficient ρ.*

PROOF: To prove Theorem 3 we shall show that any linear function $t_1 Z_{n1} + t_2 Z_{n2}$ will be normally distributed in the limit if t_1 and t_2 are not both zero. It will then follow[9] that the theorem is true.

If we define $\hat{U}_N$ to be the sequence whose elements are

$$\hat{x}_{\nu N} = \frac{t_1(x_{\nu N1} - \bar{x}_{N1})}{\mu_{2N1}^{1/2}} + \frac{t_2(x_{\nu N2} - \bar{x}_{N2})}{\mu_{2N2}^{1/2}},$$

then the arithmetic mean of $\hat{U}_N$ is zero. Let

$$\hat{z}_n = \sum_j c_{jn} \hat{x}'_{jN},$$

and let

$$\hat{Z}_n = \frac{\hat{z}_n - E\hat{z}_n}{\sigma_{\hat{z}n}}.$$

Then, it is readily verified that

$$\hat{Z}_n = \frac{t_1 Z_{n1} + t_2 Z_{n2}}{\sigma_{t_1 Z_{n1} + t_2 Z_{n2}}}.$$

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[8] The generalization holds for any finite number of components but, to simplfy the discussion, is stated for two components only. The method used is due to H. Cramér, *Random Variables and Probability Distributions*, Cambridge University Press, London, 1937, p. 105.

[9] H. Cramér, *Random Variables and Probability Distributions*, Cambridge University Press, London, 1937, p. 105.

Consequently, to prove that $t_1Z_{n1} + t_2Z_{n2}$ has a normal limiting distribution, we need to verify that the sequence U_N satisfies the condition W if U_{N1} and U_{N2} do.

The moments of $\hat{U}_N$ are

$$\hat{\mu}_{rN} = \frac{1}{N} \sum_{\nu} \hat{x}_{\nu N}^{r},$$

so that

$$\hat{\mu}_{2N} = t_1^2 + t_2^2 + 2t_1t_2\rho_N,$$

where ρ_N has the usual form of the correlation coefficient. Furthermore, using the binomial expansion, we have

$$\hat{\mu}_{rN} = \sum_{\alpha=0}^{r} C_{\alpha}^{r} \frac{t_1 t_2^{r-\alpha} \mu_{\alpha,r-\alpha N}}{\mu_{2N1}^{\alpha/2} \mu_{2N2}^{(r-\alpha)/2}}, \tag{4.4}$$

where

$$\mu_{\alpha,r-\alpha N} = \frac{1}{N} \sum_{\nu} (x_{\nu N1} - \bar{x}_{N1})^{\alpha}(x_{\nu N2} - \bar{x}_{N2})^{r-\alpha}.$$

Then, by the Cauchy-Schwarz inequality we have

$$\left| \sum_{\nu} (x_{\nu N1} - \bar{x}_{N1})^{\alpha}(x_{\nu N2} - \bar{x}_{N2})^{r-\alpha} \right| \leq \left[\sum_{\nu} (x_{\nu N1} - \bar{x}_{N1})^{2\alpha} \cdot \sum_{\nu} (x_{\nu N2} - \bar{x}_{N2})^{2r-2\alpha}\right]^{\frac{1}{2}},$$

so that

$$| \mu_{\alpha,r-\alpha N} | \leq \mu_{2\alpha,N1}^{1/2} \mu_{2r-2\alpha,N2}^{1/2},$$

and using condition W for U_{N1} and U_{N2}, we have

$$\mu_{2\alpha,N1} \leq \mu_{2N1}^{\alpha}\lambda(N), \qquad \mu_{2r-2\alpha N2} \leq \mu_{2N2}^{r-\alpha}\lambda(N).$$

Hence, substituting in (4.4) we see that

$$\sup | \mu_{rN} | < \infty.$$

Hence the sequence U_N satisfies the condition W for all t_1 and t_2, and Theorem 3 is proved.

From Theorem 3, it then follows that the theorems on the limiting distributions of moments, product moments and functions of moments[10] are valid for sampling from finite universes, at random without replacement.

[10] The most important of these theorems are given in H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press, Princeton, 1940, sections 28.2–28.4, pp. 364–367.