population are independent, the probability that a sample is within control limits on both charts is the product of the probabilities: P_1P_2. Thus the probability that a sample be outside of control limits on either chart is $1 - P_1P_2$.

The probability of the largest and smallest values both lying in the interval from $-c$ to c is: $P_3 = \Pr(-c < S, L < c) = \left[\int_{(-c-a)/\sigma}^{(c-a)/\sigma} \varphi(t)\, dt\right]^n$. Values of this expression with lower limit $-\infty$ are given in table XXI of [8] for sample of sizes 3, 5, and 10. For the purpose of comparing the charts, we choose c so that the probabilities of Type 1 errors are equal, that is: $1 - P_1P_2 = 1 - P_3$ or $P_1P_2 = P_3$ when the mean is zero and the standard deviation unity. Substituting in this equation and solving, we find: $F(c) = 0.5 + 0.5\ (.9973P_1)^{1/n}$, where $F(x) = \int_{-\infty}^{x} \varphi(t)\, dt$. For $n = 3$, $c = 2.99$ and for $n = 5$, $c = 3.15$.

Comparing P_1P_2 with P_3 when the true values are a and σ will then show the relative power of the $\bar{X}$ & R charts and the L & S chart for detecting lack of control.

Finally the charts are compared by finding the number (N_1 for the $\bar{X}$ & R charts and N_3 for the L & S chart) of samples which will detect lack of control with a .99 probability under the conditions given above. This is done by finding the smallest integer which satisfies the following inequalities: $(P_1P_2)^{N_1} < .01$ and $P_3^{N_3} < .01$. As may be seen from table II, under most conditions, the L & S chart is nearly as good as the $\bar{X}$ & R charts for detecting lack of control.

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SUFFICIENCY, TRUNCATION AND SELECTION[1]

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1. Summary. The fact that the mean and variance were sufficient statistics for a univariate normal distribution truncated at a fixed point was known to

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Fisher by 1931 [2]. Hotelling [3] has recently observed the corresponding fact for the truncated multivariate normal distribution.

It is the aim of this note to point out that these are special cases of a general result, namely: *If a family of distributions admits a set of sufficient statistics, then the family obtained by truncation to a fixed set, or by fixed selection, also admits the SAME set of sufficient statistics.*

2. Representation. The basic formal results about sets of sufficient statistics are due to Fisher [1], whose arguments, with obvious modifications, establish that families of distributions satisfying the usual conditions have sufficient statistics. The converse was established by Koopman [4] for a reasonably wide class of families.

The usual condition can be easily handled and given wide application by representing the family of distributions in a form suggested to the author by Rubin, and ascribed by him to Cramér, namely:

$$dF(x \mid \theta) = c(\theta)f(x \mid \theta)\, d\mu(x),$$

where x is a possibly multidimensional chance quantity (i.e. random variable), θ is a possibly multidimensional parameter, $c(\theta)$ is a positive real function of θ which serves to normalize the distribution, $f(x \mid \theta)$—the relative probability density—is a non-negative real function of x and θ, and $\mu(x)$ is a positive measure function. In this representation the natural and sufficient condition that $\{h_i(x)\}$ are a set of sufficient statistics for θ is the existence of functions $a_i(\theta)$ such that (cf. Koopman [4])

$$\frac{\partial \log f(x \mid \theta)}{\partial \theta} = \sum_i a_i(\theta)h_i(x). \tag{1}$$

When θ is a vector, the derivative is to be interpreted as the gradient (a vector) and the $a_i(\theta)$ are to be vector-valued functions of θ. We notice that this condition concerns only the *relative* density function.

3. Proof of result. Suppose the family $F(x \mid \theta)$ is truncated onto a Borel set E, this means that

$$\Pr\{x \text{ in } E_1 \mid F(x \mid \theta) \text{ truncated to } E\} = \frac{\Pr\{x \text{ in } E \cap E_1 \mid F(x \mid \theta)\}}{\Pr\{x \text{ in } E \mid F(x \mid \theta)\}}.$$

If $\phi_E(x)$ is the characteristic function of E, which is $=1$ for x in ϵ and $=0$ otherwise, and if

$$k(\theta) = \Pr\{x \text{ in } E \mid F(x \mid \theta)\} = \int_E dF(x \mid \theta),$$

then the probability element of $F(x \mid \theta)$ truncated to E is

$$c(\theta)/k(\theta)f(x \mid \theta)\phi_E(x)\, d\mu(x) = c'(\theta)f(x \mid \theta)\, d\nu(x),$$

where $c'(\theta) = c(\theta)/k(\theta)$ and $d\nu(x) = \phi_E(x)\, d\mu(x)$. Truncation has not changed the relative density function, and the result follows from the form of (1).

Next suppose that, instead of accepting values with probability one in E and with probability zero outside E, we select according to a fixed Borel function $\phi(x)$, the chance of accepting a value x being $\phi(x)$. The new family of distributions has the same sufficient statistics for the same reason.

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ON A PROBABILITY DISTRIBUTION

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1. Introduction. The problem treated is that of generalizing the Bernouilli distribution to the case where the probability of success is not constant from trial to trial but depends on the number of previous successes. The case where the probability of an event depends on the number of trials is easily handled and is not the case treated here. Several special cases of such a distribution have been worked out at one time or another. (E.g. C. C. Craig found the solution for one such special case and thus called the author's attention to the problem.)

The solution involves the Newton divided difference expansion of powers in a form which can be utilized for computation if the number of trials is not too large. In the case where the probabilities on a single trial are small an approximation, (similar to that of the Poisson distribution to the Bernouilli distribution) can be found.

Applications can obviously be made to urn schema in which black balls are replaced, but white balls are removed. Similarly, applications can be made to the distribution of the number of plants in a given area.

2. Solution of the problem. Specifically the problem is as follows: "What is the probability that in n trials of an event it will occur x times presuming that the probability of the event on a given trial depends only on the number of previous successes?" Denote by $P(n, x)$ the probability of x successes in n trials and by p_x the probability of the event after x previous successes. As