ABSTRACTS OF PAPERS

(Presented at the Berkeley Meeting of the Institute, June 16–18, 1949)

1. Extension of a Theorem of Blackwell. E. W. BARANKIN, University of California, Berkeley.

It is proved that Blackwell's method of uniformly improving the variance of an unbiased estimate by taking the conditional expectation with respect to a sufficient statistic, is, in fact, similarly effective on every absolute central moment of order $s \geqq 1$. The method leads to finer detail concerning the relationship between an estimate and its thus derived one. (This paper was prepared with the partial support of the Office of Naval Research.)

2. On the Existence of Consistent Tests. AGNES BERGER, Columbia University., New York.

Let $\mathfrak{M}(\mathfrak{B})$ denote the space of all probability-measures defined over a common Borel-field $\mathfrak{B}$. Let $\{m\} = M$, $\{m'\} = M'$ be two disjoint subsets of $\mathfrak{M}(\mathfrak{B})$ and let H_0 (H_1) be the hypothesis stating that the unknown distribution is in M (M'). In Neyman's terminology H_0 can be consistently tested against H_1 if to any preassigned $\epsilon > 0$ there exists an integer n and a critical region in the product-space of n independent observations such that the probabilities of the errors of the first and second kind corresponding to this region are simultaneously smaller than ϵ. A sufficient condition which for a certain type of consistent test is also necessary is established. The condition is satisfied whenever the disjoint sets M and M' are closed and compact with respect to a certain suitable topology introduced on $\mathfrak{M}(\mathfrak{B})$. Thus for instance H_0 can be consistently tested against H_1 if M and M' contain only a finite number of measures or if the measures in M resp. M' depend continuously on a parameter ranging over a closed and bounded subset of some Euclidean space.

3. Effect of Linear Truncation in a Multinormal Population. Z. WILLIAM BIRNBAUM, University of Washington, Seattle.

Let $(X, Y_1, Y_2, \cdots, Y_{n-1})$ have a non-singular n-dimensional normal probability density $f(X, Y_1, Y_2, \cdots, Y_{n-1})$ for which all parameters are given, and let $\varphi(X, Y_1, Y_2, \cdots, Y_{n-1})$ be the probability density obtained from f by truncation along a given hyperplane: $\varphi = Cf$ for $a_1Y_1 + \cdots + a_{n-1}Y_{n-1} \leqq aX + b$, $\varphi = 0$ elsewhere. What is the marginal distribution of X for this truncated distribution? This question can be answered by using a set of tables with only two parameters. These tables make it also possible to solve problems such as: determine the plane of truncation so that the marginal distribution of X has certain required properties. (This paper was prepared under the sponsorship of the Office of Naval Research.)

4. Statistical Problems in the Theory of Counters. (Preliminary Report). COLIN R. BLYTH, University of California, Berkeley.

The assumptions made about counter action and distribution of incident particles are the same as those of B. V. Gnedenko [On the theory of Geiger-Müller counters, *Journ. Exper. i Teor. Phiz*, Vol. 11 (1941)]. The distribution of the number X of particles registered during a given time $(0, t)$ is found explicitly, in terms of the density $a(v)$ of incident particles at time v. The problem considered is that of estimating the parameters of $a(v)$. For the special case $a(v) = a$, the distribution of X reduces to $P\{X = x\} = a^x(t - x\tau)^x$ exp

$\{-a(t - x\tau)\}/x! + \exp\{-a(t - x\tau)\}\Sigma_{i=0}^{x-1} a^i[t - x\tau]^i/i! - \exp\{-a[t - (x - 1)\tau]\}\Sigma_{i=0}^{x-1} a^i[t - (x - 1)\tau]^i/i!$ for $x = 1, 2, \cdots, s = \left[\frac{t}{\tau}\right]$; $P\{X = 0\} = e^{-at}$; $P\{X = s + 1\} = 1 - \exp\{-a(t - s\tau)\} \sum_{i=0}^{s} a^i[t - s\tau]^i/i!$; $P\{X > s + 1\} = 0$. This distribution has been found in another problem by J. Neyman [*On the problem of estimating the number of schools of fish*, submitted to Statistical Series, Univ. of Calif. press]. For this special case the maximum likelihood estimate $\hat{a}$ of a is found to be given by $\hat{a}\tau \exp(\hat{a}\tau) = \{1 + \tau/(t - x\tau)\}^x x\tau/(t - x\tau)$. If $\tau/(t - x\tau)$ is small, as will usually be the case, $\hat{a}$ will be close to the estimate $x/(t - x\tau)$ usually used for a.

5. **Some Two-Sample Tests.** Douglas G. Chapman, University of California, Berkeley.

Let X, Y be random variables normally distributed with means ξ, η, variances σ_1, σ_2 respectively. The two sample procedure formulated by Stein to obtain a test with power independent of σ, for the hypothesis $\eta = \xi_0$ is used here to determine a test for the hypothesis $\frac{\xi}{\eta} = r$ (r any pre-assigned real number). The size and power of this test are independent of σ_1 and σ_2. The two sample procedure may be extended to the more general case of testing the hypothesis of equality of means of several normal populations, the variances being unknown. Approximate tests are obtained for this case. Finally it is shown that this two sample procedure can be used to select that normal population, of several, with the greatest mean: the rule of selection having a preassigned level of accuracy. (This paper was prepared with the partial support of the Office of Naval Research.)

6. **Minimum Variance in Non-Regular Estimation.** R. C. Davis, U. S. Naval Ordnance Test Station, Inyokern.

The Cramér-Rao inequality for the minimum variance of a regular estimate of an unknown parameter of a probability distribution is extended to a broad class of non-regular types of estimation. The theory is developed only for the case in which a probability density function and a sufficient statistic for the unknown parameter exist. For every non-regular estimation problem included in the above class, it is proved that there exists a unique unbiased estimate which attains minimum variance, and a method is given for obtaining the sample estimate. Examples are given; such as, the rectangular distribution, a class of truncated distributions, etc.

7. **Auxiliary Random Variables.** Mark W. Eudey, California Municipal Statistics, Inc., San Francisco.

In testing hypotheses concerning discontinuous random variables it is not possible to find regions of arbitrary size, and so if we compare two critical regions, selection between them on the basis of the usual criteria of the Neyman–Pearson theory of testing hypotheses may be confused by the difference in their sizes. This difficulty may be avoided by allowing the statistician to use a mixed strategy in such cases, and make his decision to accept or reject the hypothesis depend upon an independent auxiliary random variable. For example, if K is a binomial variable, and U has a uniform distribution $(0, 1)$, then $Z = K + U$ may be used to test hypotheses concerning the binomial parameter, and regions of any size may be found. For the binomial case this procedure leads to a class of uniformly most powerful tests for one-sided alternatives, and to uniformly most powerful unbiased tests for twosided alternatives. Similar results are obtained for other common discontinuous variables, and the same device may be used in considering confidence regions and decision functions for such variables. (This paper was prepared with the partial support of the Office of Naval Research.)

8. Estimation in Truncated Samples. Max Halperin, The Rand Corporation, Santa Monica, California.

A death process is considered which starts with n individuals of zero age, each following the mortality law, $f(x, \theta)$. That is,

$$F(t) = Pr\ \{\text{Age at death} < t\} = \int_0^t f(x, \theta)\, dx,$$

where $f(x, \theta)$ is a probability density. We suppose we truncate the process at a fixed time, T, and wish to estimate θ when

a) individuals who die are not replaced, and
b) individuals who die are replaced by individuals of zero age following the mortality law, $f(x, \theta)$.

In both cases, it is found that, under mild conditions, estimation by Maximum Likelihood gives optimum estimates. The estimates are best in the sense of being asymptotically normally distributed and of minimum variance for large samples.

The proofs are given for the case of a single parameter, but can be extended to the multiparameter case. Examples are given.

9. Some Problems in Point Estimation. J. L. Hodges, Jr. and E. L. Lehmann, University of California, Berkeley.

Some point estimation problems are considered in the light of Wald's general theory. It is shown that when the loss function is convex, one may restrict consideration to nonrandomized estimates based on sufficient statistics. Minimax estimates are obtained in a number of cases connected with the binomial and hypergeometric distributions, and with some non-parametric problems. Some prediction problems are also considered. (This paper was prepared with the partial support of the Office of Naval Research.)

10. Completeness in the Sequential Case. E. L. Lehmann and C. Stein, University of California, Berkeley.

Recently, in a series of papers, Girshick, Mosteller, Savage and Wolfowitz have considered the uniqueness of unbiased estimates depending only on an appropriate sufficient statistic for sequential sampling schemes of binomial variables. A complete solution was obtained under the restriction to bounded estimates. This work, which has immediate consequences with respect to the existence of unbiased estimates with uniformly minimum variance, is extended here in two directions. A general necessary condition for uniqueness is found, and this is applied to obtain a complete solution of the uniqueness problem when the random variables have a Poisson or rectangular distribution. Necessary and sufficient conditions are also found in the binomial case without the restriction to bounded estimates. This permits the statement of a somewhat stronger optimum property for the estimates, and is applicable to the estimation of unbounded functions of the unknown probability.

11. The Ratio of Ranges. Richard F. Link, University of Oregon, Eugene.

The distribution of the ratio of two ranges from independent samples drawn from a normal population is given analytically for n_1 and $n_2 \leqq 3$. A table of percentage values, R, is given for $\alpha = .005, .01, .025, .05, .10$ and for all combinations of n_1 and n_2 up to 10, where $\alpha = \Pr(w_1/w_2 > R)$ and w_1 and w_2 are the observed ranges. (This paper was prepared under the sponsorship of the Office of Naval Research.)

12. Some Problems Arising in Plant Selection and the Use of Analysis of Variance. Stanley W. Nash, University of California, Berkeley.

The yields of many (m) varieties are compared in a field trial. A few varieties having the highest and lowest yields in this trial are selected for further testing. What chance is there that the first trial will give a significant result, the second trial not? Let ξ_i denote the true mean yield of the ith variety, and assume that the ξ_i are themselves normally, independently distributed with variance σ_1^2. Let P_k ($k = 1, 2$) denote the probability of a significant result in the kth trial, using the F-test. For fixed $\sigma_1^2 > 0$, $\lim_{m\to\infty} P_1 = 1$. (See Nash, *Annals of Math. Stat.*, Vol. 19 (1948), p. 434.) Now let $\sigma_1^2 > 0$ take on a decreasing sequence of values as m increases. If $\frac{1}{\sigma_1^2 g(m)} = 0\left(\frac{E(F)}{\sigma_F}\right)$, then $\lim_{m\to\infty} P_1 = 1$. Here $1 + \sigma_1^2 g(m) = \frac{E(\text{numerator of } F)}{\sigma_0^2\ (= \text{error variance})}$. Also $\lim_{m\to\infty} P_2 < 1$ if and only if $\sigma_1^2 = 0\left(\frac{1}{\sqrt{\log m}}\right)$. For $\sigma_1^2 = 0\left(\frac{1}{\sqrt{\log m}}\right)$, $\lim_{m\to\infty} P_2 = \alpha$, the level of significance used. Thus, corresponding to any m, however large, one can find values of σ_1^2 for which the chances are considerable (or even approaching $1 - \alpha$), that the two field trials will give opposite conclusions when the F-test is used.

13. Asymptotic Properties of the Wald-Wolfowitz Test of Randomness. Gottfried E. Noether, Columbia University, New York.

Let $a_1, \cdots, a_n$ be observations on the chance variables $X_1, \cdots, X_n$. Wald and Wolfowitz (*Annals of Math. Stat.*, Vol. 14 (1943), pp. 378–388) have shown how the statistic $R_h = \sum_{i=1}^n x_i x_{i+h}$, $(x_{n+j} = x_j)$, can be used to test the null hypothesis that the X_i, $(i = 1, \cdots, n)$, are independently and identically distributed by considering the distribution of R_h in the subpopulation of all permutations of the a_i. In the present paper it is shown that when the null hypothesis is true this distribution of R_h is asymptotically normal provided $\sum_{i=1}^n (a_i - \bar{a})^r / [\sum_{i=1}^n (a_i - \bar{a})^2]^{r/2} = 0[n^{(2-r)/4}]$, $(r = 3, 4, \cdots)$, a condition which is satisfied with probability 1 if the a_i are independent observations on the same chance variable X having positive variance and a finite absolute moment of order $4 + \delta$, $(\delta > 0)$. Conditions are given for the consistency of the test based on R_h when under the alternative hypothesis observations are drawn independently from changing populations. In particular a downward trend and a regular cyclical movement are considered, both for ranks and original observations. For the special case of a regular cyclical movement of known length the asymptotic relative efficiency of the rank test with respect to the test performed on original observations is found. It is shown that when using ranks, R_h is asymptotically normal under the alternative hypothesis provided $\lim\inf_{n\to\infty} \text{var}(n^{-5/2}R_h) > 0$. This asymptotic normality of R_h is used to compare the asymptotic power of the R_h-test with that of the Mann T-test (*Econometrica*, Vol. 13 (1945), pp. 245–259) for the case of a downward trend.

14. On the Similar Regions of a Class of Distributions. Stefan Peters, University of California, Berkeley.

The class of distributions considered is essentially the class of those distributions of n variables which, by a suitable transformation of the variables and the parameter, can be transformed into distributions defined in the whole R_n for which the parameter is a location parameter. These regions satisfy a certain partial differential equation. The transformed distributions of the variables $y_1, \cdots y_n$ and parameter τ possess a class D_1 of similar regions with respect to τ which can be defined as the smallest additive class of regions which includes all regions defined by

$$g[(y_1 - y_n), \cdots, (y_n + y_n)] \geqq C$$

where g is a continuous function. The class D_1 does not exhaust all similar regions. There exists among the regions of class D_1 one which is most powerful for testing a given additional parameter σ. If there exists among all similar regions a most powerful region for testing σ, then that region will be the most powerful region of class D_1.

15. Some Problems in Sequential Analysis. Charles M. Stein, University of California, Berkeley.

Wald's fundamental identity for cumulative sums is extended to dependent random variables. The first derivative of this at the origin is equivalent to a result of Wolfowitz (*Annals of Math. Stat.*, Vol. 18 (1947), p. 228, Th. 7.4). Higher derivatives of this at the origin can also be obtained from linear combinations of Wolfowitz's result applied to suitable products of the original random variables. These equations yield approximate *OC* and *ASN* curves for probability-ratio tests for a simple hypothesis against a single alternative concerning some of the more usual stationary Markoff chains. Bounds for the amount by which the *ASN* exceeds that of the most efficient test are also obtained. The results are applied in particular to random variables taking on only the values 0, 1 with conditional probabilities depending only on a finite number of the preceding observations. The case of linear dependence of normal random variables with fixed conditional variance is also considered.

16. Some Aspects of Links Between Prediction Problems and Problems of Statistical Estimation. Erling Sverdrup, University of Oslo.

A prediction is not taken as a probability statement about additional observations of the random variable already observed. It is presumed that the statistical interpretation of the sample will result in some action influencing the random variable subject to prediction. The probability distribution of this random variable is given for each of an a priori class of probability functions for the observed random variable and for each of a class of possible actions. "Utility" as a function of the random variable to be predicted and of the action is defined. It is shown that the problem of which action to take in order to maximize expected utility is identical with a problem of statistical inference with a uniquely defined weight function in the Wald sense. It is further shown that this procedure is adaptable to stochastic processes of a general type and this provides a means of connecting the theory of stochastic processes with the theory of statistical inference. Some examples are given to illustrate the general theory.

17. Some Large Sample Tests for the Median. John E. Walsh, The Rand Corporation, Santa Monica, California.

Consider a large number of independent observations from continuous populations with a common median. Sone non-parametric large sample tests for the population median are presented which are based on either two or three order statistics of the sample. If all the populations are symmetrical, these tests are equal-tailed with specified significance level α. If the observations are a sample from a normal population, these tests have high power efficiencies. Some tests based on three order statistics are developed which also have significance level α if all the populations are not symmetrical; however, in this case the resulting test is one-tailed instead of equal-tailed. Using these tests for situations where the populations are believed to be symmetrical furnishes a safety factor with respect to Type I error. Tests are presented for the special case where each population is either symmetrical or skewed in a specified direction. If the populations are not symmetrical the significance level distribution is $.4\alpha$ to one tail and $.6\alpha$ to the other, rather than $.5\alpha$ to each tail. Also some non-parametric large sample tests of whether a sample is from a symmetrical population are derived. These tests are based on three order statistics of the sample and have bounded significance levels.

18. Continuous Sampling Plans from the Risk Point of View. ZIVIA S. WURTELE, Stanford University, California.

The quality of a lot can be improved by a screening process whereby the defective items found during inspection are replaced by non-defective items. The type of sampling plan adopted will generally depend upon the cost of inspecting items, the number of defective items in the lot prior to inspection, and the loss due to defective items remaining in the lot after inspection. The loss if the lot is accepted after d defectives are found in a sample of n items is equal to $c(n) + h(D)$ where D is the number of defectives left in the lot and $c(n)$ is the cost of inspecting n items. An inspection procedure S is defined by a set of stopping points $\{(d, n)\}$. Let $r(p, S)$ be the expected loss if p is the probability of a defective and the procedure S is used. It is assumed that the lot is obtained from a binomial population. For any a priori distribution $F(p)$, a Bayes procedure is one which minimizes the expected risk,

$$\int_0^1 r(p,S)dF(p).$$

A systematic method of obtaining Bayes solutions exists, but the computations are formidable. Under fairly general conditions the Bayes solutions are shown to be multiple sampling plans, in which the size of the ith sample depends upon the number of defectives in the $(i - 1)$st sample. In particular, if the production is in a state of statistical control, a Bayes solution is a fixed sample size. It is also shown that for most reasonable loss functions, there exists no mini-max procedure which is uniformly better than the trivial one; namely, the Bayes procedure if $p = 1$.