Substituting (34) in (32), and equating coefficients of like powers of (x, y), we obtain the recursion formulae

$$(35)\quad \sum_{j+k=n} B_{ij}B_{0k}[j][2k-j+1] = \sum_{j+k=n-1} B_{i+1,j}B_{0k}[i+1][j-k]; \qquad i: 0, 1, \cdots.$$

From (10), it is readily verified that $B_{i0} = 0$ for $i \neq 0$, so that equations (35) give solutions for the B_{ij} in terms of the B_{0k}. These solutions are of interest since they show a one-to-one correspondence between the functions $G(0, y)$ and $G(x, y)$, for $(x, y) \epsilon [R \cap S]$.

NUMERICAL INTEGRATION FOR LINEAR SUMS OF EXPONENTIAL FUNCTIONS

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1. Introduction. The methods of numerical integration going by the names trapezoidal rule, Simpson's rule, Weddle's rule, and the Newton-Cotes formulae are of the type

$$(1)\qquad \int_{-1}^{1} f(x)\,dx \simeq \sum_{i=0}^{n} \lambda_{in} f(x_{in})$$

where the abscissae $\{x_{in}\}$ are uniformly distributed on a finite interval, chosen as $(-1, 1)$ for convenience,

$$(2)\qquad x_{in} = -1 + \frac{2i}{n}, \qquad i = 0, 1, 2, \cdots, n,$$

and where the set of constants $\{\lambda_{in}\}$ depend on the name of the rule and the value of n but not on the function $f(x)$. Throughout this note all abscissae will be assumed to be uniformly distributed on $(-1, 1)$ unless the contrary is explicitly stated.

Since correspondence relation (1) involves $(n + 1)$ constants $\{\lambda_{in}\}$, it might be possible to choose $(n + 1)$ arbitrary functions $g_j(x)$, $j = 0, 1, 2, \cdots, n$, and require that the set $\{\lambda_{in}\}$ be the solution, if such exists, of the $(n + 1)$ simultaneous linear equations

$$(3)\qquad \int_{-1}^{1} g_j(x)\,dx = \sum_{i=0}^{n} \lambda_{in} g_j(x_{in}), \qquad j = 0, 1, 2, \cdots, n.$$

Indeed, the selection

$$(4)\qquad g_j(x) = x^j, \qquad j = 0, 1, 2, \cdots, n,$$

will give a set of $(n + 1)$ simultaneous equations of form (3) and the solution $\{\lambda_{in}\}$ is the set of Newton-Cotes weights for that value of n. The numerical evaluation

[1] This work was performed with the financial support of the Office of Naval Research of the Navy Department.

of $\{\lambda_{in}\}$ is best accomplished by other and more sophisticated methods, however.[2]

Because of linearity in both the integral and the finite summation, once the constants $\{\lambda_{in}\}$ have been determined for a specific set of functions $\{g_j(x)\}$, correspondence relation (1) is exact for any linear combination of that fundamental set. Thus, for example, for the fundamental set (4), correspondence relation (1) with the appropriate values $\{\lambda_{in}\}$ is exact for all polynomials of degree less than or equal to n.

Although tradition favors the set of functions (4), there is nothing compelling about such a selection. Indeed, two other possible choices might be

$$g_j(x) = e^{jx}, \qquad j = 0, 1, 2, \cdots, n, \tag{5}$$

and

$$g_j(x) = e^{jx}, \qquad j = -m, -m + 1, \cdots, 0, 1, \cdots, m - 1, m; n = 2m. \tag{6}$$

These choices would seem to be appropriate whenever numerical methods are being applied to exponential growth curves or exponential decay curves.

2. Use of the basic set $g_j(x) = e^{jx}$. If integration relation (1) be made exact for the set $\{e^{jx}\}$, $j = 0, 1, \cdots, n$ with evenly spaced x abscissae, the set (3) of $(n + 1)$ simultaneous linear equations in the unknowns $\{\lambda_{in}\}$, $i = 0, 1, \cdots, n$ is obtained. Call the solution of this system $\{a_{in}\}$, solution values for $n = 1, 2, 3, 4, 5, 6$ are tabulated below.

For the symmetric case where integration relation (1) is made exact for $\{e^{jx}\}$, $j = -m, -m + 1, \cdots, m - 1, m; n = 2m$, a similar but different set of linear equations (3) results for the unknowns $\{\lambda_{in}\}$. Call the solution of this system $\{b_{in}\}$. As implied above, only even values of n are used in order to preserve the symmetry, and values of $\{b_{in}\}$ are tabulated below for $n = 2, 4, 6$.

$n = 1,$	$a_{01} =$ 1.31303 5285	
	$a_{11} =$ 0.68696 4715	
$n = 2,$	$a_{02} =$ 0.21805 032^{+}	$b_{02} =$ 0.32260 623^{-}
	$a_{12} =$ 1.49780 742	$b_{12} =$ 1.35478 755
	$a_{22} =$ 0.28414 226^{-}	$b_{22} =$ 0.32260 623^{-}
$n = 3,$	$a_{03} =$ 0.51324 284	
	$a_{13} =$ 0.22445 055	
	$a_{23} =$ 1.08155 527	
	$a_{33} =$ 0.18075 134	
$n = 4,$	$a_{04} =$ −0.13716 639^{+}	$b_{04} =$ 0.15048 171
	$a_{14} =$ 1.40098 548	$b_{14} =$ 0.73243 318

[2] Whittaker and Robinson, *The Calculus of Observations*, 4th Edition, (1946), London, pp. 152–156.

	$a_{24} = -0.30895\ 914$	$b_{24} = 0.23417\ 022$
	$a_{34} = 0.91710\ 903$	$b_{34} = 0.73243\ 318$
	$a_{44} = 0.12803\ 103^{-}$	$b_{44} = 0.15048\ 171$
$n = 5,$	$a_{05} = 0.68919\ 3$	
	$a_{15} = -1.07644\ 3$	
	$a_{25} = 2.12534\ 6$	
	$a_{35} = -0.63595\ 6$	
	$a_{45} = 0.79933\ 8$	
	$a_{55} = 0.09852\ 18$	
$n = 6,$	$a_{06} = -0.83607$	$b_{06} = 0.09443\ 5$
	$a_{16} = 3.54128$	$b_{16} = 0.53464\ 7$
	$a_{26} = -3.88102$	$b_{26} = 0.01139\ 3$
	$a_{36} = 3.32254$	$b_{36} = 0.71905\ 0$
	$a_{46} = -0.94685$	$b_{46} = 0.01139\ 3$
	$a_{56} = 0.72075$	$b_{56} = 0.53464\ 7$
	$a_{66} = 0.07937\ 5^{+}$	$b_{66} = 0.09443\ 5$

The computing service of the Institute for Numerical Analysis has supplied the author with most of the coefficients tabulated above.

3. Estimates of the error term. The choices of the coefficients $\{a_{in}\}$ and $\{b_{in}\}$ are such that integration relation (1) is exact whenever

$$f(x) = A_0 + A_1 e^{x} + \cdots + A_n e^{nx} \quad \text{and} \quad \lambda_{in} = a_{in}, \tag{7}$$

and whenever

$$f(x) = B_{-m} e^{-mx} + B_{-m+1} e^{-(m-1)x} + \cdots + B_0 + \cdots + B_m e^{mx} \text{ and } \lambda_{in} = b_{in}. \tag{8}$$

When $f(x)$ is not of these prescribed forms, the error in using correspondence (1) may be of some importance. By making the transformation

$$u = e^{x}, \qquad f(x) = f(\log u) = g(u) \tag{9}$$

integration relation (1) becomes

$$\int_{e^{-1}}^{e} g(u) \frac{du}{u} \simeq \sum_{i=0}^{n} \lambda_{in} g(u_{in}) \tag{10}$$

where the $\{u_{in}\}$ are not evenly distributed. By approximating $g(u)$ by its Taylor's series with a remainder term, the following expressions for the error in using correspondence (1) can be obtained:

Using the coefficients $\{a_{in}\}$,

$$\text{Error} \leq \frac{\left(\dfrac{e^2 - 1}{2e}\right)^{n+1}}{(n+1)!} \left[2 + \sum_{i=0}^{n} |a_{in}|\right] \left[\max_{-1 \leq x \leq 1} \left(e^{-x} \frac{d}{dx}\right)^{n+1} f(x)\right] \tag{11}$$

and, using the coefficients $\{b_{in}\}$,

$$\text{(12)} \quad \text{Error} \leq \frac{\left(\frac{e^2-1}{2e}\right)^{2m+1}}{(2m+1)!}\left[\frac{e^m - e^{-m}}{m} + \sum_{i=0}^{2m} \frac{|b_{i,2m}|}{e^{mx_{i,2m}}}\right] \cdot \left[\max_{-1 \leq x \leq 1} \left(e^{-x}\frac{d}{dx}\right)^{2m+1} e^{mx} f(x)\right].$$

Neither of these error expressions can be said to be very practical in actual computation, and neither appears suitable for establishing convergence properties of the type

$$\text{(13)} \quad \lim_{n\to\infty} \sum_{i=0}^{n} \lambda_{in} f(x_{in}) = \int_{-1}^{1} f(x)\, dx.$$

However, both (11) and (12) reduce to zero when $f(x)$ is of the form prescribed by (7) or (8) respectively.

4. Numerical examples. As illustrative numerical examples, the case $n = 4$ was selected and several typical functions were integrated approximately by the positive power exponential rule, the symmetrical exponential rule and the Newton-Cotes formula,

$$\int_{-1}^{1} f(x)\, dx = \tfrac{1}{45}[7f(-1) + 32f(-\tfrac{1}{2}) + 12f(0) + 32f(\tfrac{1}{2}) + 7f(1)].$$

Values of $\{a_{i4}\}$ and $\{b_{i4}\}$ are given in the tables in part 2. The typical functions used were x^2, e^{2x}, $1/(x + 3)$, e^{-x^2}, xe^x, x^6, and $e^{2.2x}$. The following results were obtained:

Function	*Positive Power Expontential*	*Symmetrical Exponential*	*Newton-Cotes*	*8 Decimal Approximation to Exact Value*
x^2	.5703 8827	.6671 8001	.6666 6666	.6666 6667
e^{2x}	3.6268 6044	3.6268 6041	3.6317 3108	3.6268 6041
$1/(x + 3)$	.6828 6353	.6931 5792	.6931 7460	.6931 4718
e^{-x^2}	1.4930 1396	1.4857 2754	1.4887 4582	1.4936 4827
xe^x	.7292 4338	.7353 6007	.7361 7480	.7357 5888
x^6	.0270 8487	.3238 5196	.3333 3332	.2857 1429
$e^{2.2x}$	4.0527 7287	4.0530 7585	4.0607 7415	4.0519 1379

From this tabulation, it would appear that the symmetrical exponential method compares favorably with the Newton-Cotes method for such typical functions as $1/(x + 3)$, e^{-x^2}, xe^x, x^6, and $e^{2.2x}$. Note that the choice of x^2 or e^{2x} is not really a fair choice when comparing these two methods, since Newton-Cotes is derived so as to give exactness for x^2 and the symmetrical exponential so as to give exactness for e^{2x}.