A SEQUENTIAL DECISION PROCEDURE FOR CHOOSING ONE OF THREE HYPOTHESES CONCERNING THE UNKNOWN MEAN OF A NORMAL DISTRIBUTION

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1. Introduction. In this paper a multi-decision problem is investigated from a sequential viewpoint and compared with the best non-sequential procedure available. Multi-decision problems occur often in practice but methods to deal with such problems are not yet sufficiently developed.

The problem under consideration here is a 3-decision problem: Given a chance variable which is normally distributed with known variance σ^2, but unknown mean θ, and given two real numbers $a_1 < a_2$, the problem is to choose one of the three mutually exclusive and exhaustive hypotheses

$$H_1 : \theta < a_1 \qquad H_2 : a_1 \leqq \theta \leqq a_2 \qquad H_3 : \theta > a_2 .$$

In order to select a proper sequential decision procedure, the parameter space is subdivided into 5 mutually exclusive and exhaustive zones in the following manner. Around a_1 there exists an interval (θ_1, θ_2) in which we have no strong preference between H_1 and H_2 but prefer (strongly) to reject H_3. Around a_2 there exists an interval (θ_3, θ_4) in which we have no strong preference between H_2 or H_3 but prefer (strongly) to reject H_1. For $\theta \leqq \theta_1$ we prefer to accept H_1. For $\theta_2 \leqq \theta \leqq \theta_3$ we prefer to accept H_2. For $\theta \geqq \theta_4$ we prefer to accept H_3.

The intervals (θ_1, θ_2) and (θ_3, θ_4) will be called indifference zones. The determination of these indifference zones is not a statistical problem but should be made on practical considerations concerning the consequences of a wrong decision.

In accordance with the above we define a wrong decision in the following way. For $\theta \leqq \theta_1$, acceptance of H_2 or H_3 is wrong. For $\theta_1 < \theta < \theta_2$ acceptance of H_3 is wrong. For $\theta_2 \leqq \theta \leqq \theta_3$, acceptance of H_1 or H_3 is wrong. For $\theta_3 < \theta < \theta_4$, acceptance of H_1 is wrong. For $\theta \geqq \theta_4$, acceptance of H_1 or H_2 is wrong.

The requirements on our decision procedure necessary to limit the probability of a wrong decision are investigated. Two cases are considered.

Case 1: Prob. of a wrong decision $\leqq \gamma$ for all θ.

$$\text{Case 2: } \begin{cases} \text{Prob. of a wrong decision } \leqq \gamma_1 \text{ for } \theta \leqq \theta_1 , \\ \text{Prob. of a wrong decision } \leqq \gamma_2 \text{ for } \theta_1 < \theta < \theta_4 , \\ \text{Prob. of a wrong decision } \leqq \gamma_3 \text{ for } \theta \geqq \theta_4 . \end{cases}$$

The decision procedure discussed in the present paper is not an optimum procedure since, as will be seen later, the final decision at the termination of

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experimentation is not in every case a function of only "the sample mean of *all* the observations", although the sample mean is a sufficient statistic for θ. Although the procedure considered is not optimal it is suggested for the following reasons:

1. The decision procedure can be carried out simply. In fact tables can be constructed before experimentation starts that render the procedure completely mechanical.

2. The derivation of the operating characteristic (OC) function, neglecting the excess of the cumulative sum over the boundary, is accomplished with little difficulty. In general, for other multi-decision problems it is unknown how to obtain the OC function.

3. It is believed that the loss of efficiency is not serious; i.e., the suggested sequential procedure is not far from being optimum. In this connection a non-sequential procedure is compared with this sequential procedure. The results show that, for the same maximum probability of making a wrong decision, the sequential procedure requires on the average substantially fewer observations to reach a final decision. In fact, for Case 1 noted above, if $.008 < \gamma < .1$, and if certain symmetrical features are assumed, then the fixed number of observations required by the non-sequential method is greater than the maximum of the average sample number (ASN) function taken over all values of θ.

It was found necessary in the course of the investigation to put an upper bound on the quantity $\frac{\theta_4 - \theta_3}{a_2 - a_1}$ in order that the methods used to obtain upper and lower bounds for the ASN function should give close results. This restriction, however, is likely to be satisfied in practical applications.

All formulas for ASN and OC functions which will be used in this paper will be approximation formulas neglecting the excess of the cumulative sum over the boundaries. Nevertheless, equality signs will be used in these formulas, except when additional approximations are involved.

2. Description of the Decision Procedure.[2] We shall assume that the indifference zones described above have the following properties

$$\text{(i)}\ \theta_1 < a_1 < \theta_2 \leqq \theta_3 < a_2 < \theta_4$$

$$\text{(ii)}\ \theta_1 + \theta_2 = 2a_1 \ ; \quad \theta_3 + \theta_4 = 2a_2$$

$$\text{(iii)}\ \theta_2 - \theta_1 = \theta_4 - \theta_3 = \Delta \text{ (say)}.$$

[2] A similar decision procedure was used by P. Armitage [2] as an alternative to the sequential t test (with 2-sided alternatives). The form used there is more restricted as he considers only the case $\theta_2 = \theta_3$. Essential inequalities on the OC function are pointed out but no attempt is made to determine the complete OC and ASN functions. A closely related but somewhat different procedure for dealing with a trichotomy was suggested by Milton Friedman while he was a member of the Statistical Research Group of Columbia University. As far as the authors are aware, no results were obtained concerning the OC and ASN functions of Friedman's procedure.

Let R_1 denote the Sequential Probability Ratio Test for testing the hypothesis that $\theta = \theta_1$ against the hypothesis that $\theta = \theta_2$. We assume for the present that either the proper constants A, B in the probability ratio test are given or that they are approximated from given α, β by the relations

$$A \sim \frac{1-\beta}{\alpha} \qquad B \sim \frac{\beta}{1-\alpha}.$$

Here α and β are upper bounds on the probabilities of first and second types of errors, respectively.

Let R_2 represent the S.P.R.T. for testing the hypothesis that $\theta = \theta_3$ against the alternative that $\theta = \theta_4$. For this test we assume that (α, β, A, B) are replaced by $(\hat{\alpha}, \hat{\beta}, \hat{A}, \hat{B})$ and as above that either $\hat{A}$ and $\hat{B}$ are given or that they are approximated from given $\hat{\alpha}$, $\hat{\beta}$.

The decision procedure is carried out as follows:

Both R_1 and R_2 are computed at each stage of the inspection until

Either: One ratio leads to a decision to stop before the other. Then the former is no longer computed and the latter is continued until it leads to a decision to stop.

Or: Both R_1 and R_2 lead to a decision to stop at the same stage. In this event both computations are discontinued.

The following table gives the rule R for the decisions to be made corresponding to all possible outcomes of R_1 and R_2.

	R_1		R_2		R
If	accepts θ_1	and	accepts θ_3	then	accepts H_1
If	accepts θ_2	and	accepts θ_3	then	accepts H_2
If	accepts θ_2	and	accepts θ_4	then	accepts H_3

We shall show that acceptance of both θ_1 and θ_4 is impossible when $(\hat{A}, \hat{B}) = (A, B)$. For this purpose we need the acceptance number and rejection number formulas. (See page 119 of [1]).

Acceptance Number *Rejection Number*

$$R_1: \frac{\sigma^2}{\Delta} \log B + a_1 n < \sum_{\alpha=1}^{n} x_\alpha < \frac{\sigma^2}{\Delta} \log A + a_1 n$$

$$R_2: \frac{\sigma^2}{\Delta} \log B + a_2 n < \sum_{\alpha=1}^{n} x_\alpha < \frac{\sigma^2}{\Delta} \log A + a_2 n.$$

We shall assume *for convenience* that "between observations" R_1 is tested before R_2 and let the term "initial decision" refer to the first decision made.

Assume θ_1 and θ_4 are both accepted. Then if θ_1 is accepted initially at the mth stage

$$\sum_{\alpha=1}^{m} x_\alpha \leqq \frac{\sigma^2}{\Delta} \log B + a_1 m.$$

Since

$$\frac{\sigma^2}{\Delta} \log B + a_1 m < \frac{\sigma^2}{\Delta} \log B + a_2 m$$

it follows that θ_4 is rejected at the same stage, contradicting the hypothesis. Similarly if θ_4 is accepted initially at the mth stage, then

$$\sum_{\alpha=1}^{m} x_\alpha \geqq \frac{\sigma^2}{\Delta} \log A + a_2 m.$$

Since

$$\frac{\sigma^2}{\Delta} \log A + a_2 m > \frac{\sigma^2}{\Delta} \log A + a_1 m$$

it follows that θ_1 is rejected at the same or at an earlier stage, contradicting the assumption that the acceptance of θ_4 is an initial decision. Hence θ_1 and θ_4 cannot both be accepted.

A geometrical representation of the rule R is given in Figure 1.

R can now be described as follows: Continue taking observations until an acceptance region (shaded area) is reached or both dashed lines are crossed. In the former case, stop and accept as shown above. In the latter case stop and accept H_2.

The proof above that θ_1 and θ_4 cannot both be accepted consists of noting that a point below the acceptance line for θ_1 is already below the rejection line for θ_4 and that a point above the acceptance line for θ_4 is already above the rejection line for θ_1.

If $(\hat{A}, \hat{B}) \neq (A, B)$, a necessary and sufficient condition for the impossibility of accepting θ_1 and θ_4 is that at $n = 1$ the following inequalities should hold.

Rejection Number (of θ_1) for $R_1 \leqq$ Rejection Number (of θ_3) for R_2

and

Acceptance Number (of θ_1) for $R_1 \leqq$ Acceptance Number (of θ_3) for R_2.

In symbols

$$\frac{\sigma^2}{\Delta} \log A + a_1 \leqq \frac{\sigma^2}{\Delta} \log \hat{A} + a_2$$

and

$$\frac{\sigma^2}{\Delta} \log B + a_1 \leqq \frac{\sigma^2}{\Delta} \log \hat{B} + a_2.$$

These can be written as

$$\frac{A}{\hat{A}} \leqq e^{d\Delta/\sigma^2} \quad \text{and} \quad \frac{B}{\hat{B}} \leqq e^{d\Delta/\sigma}$$

respectively, where $d = a_2 - a_1$.

Since $\frac{d\Delta}{\sigma^2} > 0$, the above inequalities are certainly fulfilled when

$$\frac{B}{\bar{B}} \leqq 1 \quad \text{and} \quad \frac{A}{\bar{A}} \leqq 1. \tag{2.1}$$

In what follow in this paper, we shall restrict ourselves to cases where acceptance of both θ_1 and θ_4 is impossible, even if this is not stated explicitly.

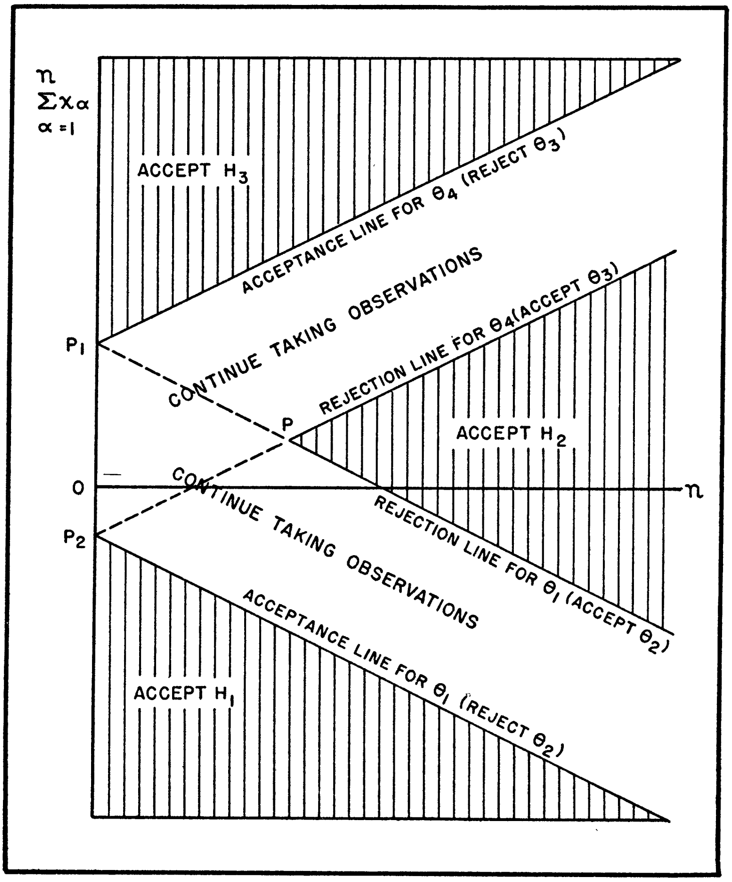

FIGURE 1

3. Derivation of OC Functions. Let $L(H_i \mid \theta, R)$ denote the probability of accepting H_i when θ is the true mean and R is the sequential rule used. Let H_{θ_i} denote the hypothesis that $\theta = \theta_i$. Since, as shown above, H_1 is accepted if and only if θ_1 is accepted, we have

$$L(H_1 \mid \theta, R) = L(H\theta_1 \mid \theta, R_1). \tag{3.1}$$

Similarly,

$$L(H_3 \mid \theta, R) = L(H_{\theta_4} \mid \theta, R_2). \tag{3.2}$$

From the fact that R_1 and R_2 each terminate at some finite stage with probability one, it follows that R will terminate at some finite stage with probability one. Hence

$$L(H_2 \mid \theta, R) = 1 - L(H_1 \mid \theta, R) - L(H_3 \mid \theta, R). \tag{3.3}$$

From pp. 50–52 of [1], the following equations are obtained.

$$L(H_1 \mid \theta, R) = L(H_{\theta_1} \mid \theta, R_1) = \frac{A^{h_1} - 1}{A^{h_1} - B^{h_1}} \tag{3.4}$$

where

$$h_1 = h_1(\theta) = \frac{\theta_2 + \theta_1 - 2\theta}{\theta_2 - \theta_1} = \frac{a_1 - \theta}{\frac{\Delta}{2}}$$

and

$$L(H_{\theta_3} \mid \theta, R_2) = \frac{\hat{A}^{h_2} - 1}{\hat{A}^{h_2} - \hat{B}^{h_2}} \tag{3.5}$$

where

$$h_2 = h_2(\theta) = \frac{\theta_4 + \theta_3 - 2\theta}{\theta_4 - \theta_3} = \frac{a_2 - \theta}{\frac{\Delta}{2}}.$$

These equations involve an approximation, as explained in [1].

Hence

$$L(H_3 \mid \theta, R) = L(H_{\theta_4} \mid \theta, R_2) = 1 - L(H_{\theta_3} \mid \theta, R_2) = \frac{1 - \hat{B}^{h_2}}{\hat{A}^{h_2} - \hat{B}^{h_2}} \tag{3.6}$$

and

$$L(H_2 \mid \theta, R) = 1 - \frac{A^{h_1} - 1}{A^{h_1} - B^{h_1}} - \frac{1 - \hat{B}^{h_2}}{\hat{A}^{h_2} - \hat{B}^{h_2}} = \frac{1 - B^{h_1}}{A^{h_1} - B^{h_1}} - \frac{1 - \hat{B}^{h_2}}{\hat{A}^{h_2} - \hat{B}^{h_2}}. \tag{3.7}$$

Since $L(H_1 \mid \theta, R) = L(H_{\theta_1} \mid \theta, R_1)$, it follows that $L(H_1 \mid \theta, R)$ is a monotonically decreasing function of θ and that

$$L(H_1 \mid -\infty, R) = 1; \qquad L(H_1 \mid \infty, R) = 0$$

$$L(H_1 \mid \theta_1, R) = 1 - \alpha; \qquad L(H_1 \mid \theta_2, R) = \beta$$

$$L(H_1 \mid a_1, R) = \frac{\log A}{\log A + |\log B|}.$$

Similarly, since $L(H_3 \mid \theta, R) = 1 - L(H_{\theta_3} \mid \theta, R_2)$, it follows that $L(H_3 \mid \theta, R)$ is a monotonically increasing function of θ and that

$$L(H_3 \mid -\infty, R) = 0; \qquad L(H_3 \mid \infty, R) = 1$$

$$L(H_3 \mid \theta_3, R) = \hat{\alpha}; \qquad L(H_3 \mid \theta_4, R) = 1 - \hat{\beta}$$

$$L(H_3 \mid a_2, R) = \frac{|\log \hat{B}|}{\log \hat{A} + |\log \hat{B}|}.$$

Since $L(H_2 \mid \theta, R) = 1 - L(H_1 \mid \theta, R) - L(H_3 \mid \theta, R)$ it follows easily from the above results that

$$L(H_2 \mid -\infty, R) = 0; \qquad L(H_2 \mid \infty, R) = 0$$

$$L(H_2 \mid \theta, R) < \alpha \quad \text{for} \quad \theta < \theta_1; \qquad L(H_2 \mid \theta, R) < \hat{\beta} \quad \text{for} \quad \theta > \theta_4$$

$$\frac{|\log B|}{\log A + |\log B|} - \hat{\alpha} < L(H_2 \mid a_1, R) < \frac{|\log B|}{\log A + |\log B|}$$

$$\frac{\log \hat{A}}{\log \hat{A} + |\log \hat{B}|} - \beta < L(H_2 \mid a_2, R) < \frac{\log \hat{A}}{\log \hat{A} + |\log \hat{B}|}$$

$$1 - \beta - \hat{\alpha} < L(H_2 \mid \theta, R) < 1 \quad \text{for} \quad \theta_2 \leqq \theta \leqq \theta_3.$$

4. Probability of Correct Decision. Denote the probability of a correct decision by $L(\theta/R)$. It is defined as follows:

Interval	*Correct Decisions*	$L(\theta \vert R)$
$\theta \leqq \theta_1$	acceptance of H_1	$L(H_1 \mid \theta, R)$
$\theta_1 < \theta < \theta_2$	acceptance of H_1 or H_2	$L(H_1 \mid \theta, R) + L(H_2 \mid \theta, R)$
$\theta_2 \leqq \theta \leqq \theta_3$	acceptance of H_2	$L(H_2 \mid \theta, R)$
$\theta_3 < \theta < \theta_4$	acceptance of H_2 or H_3	$L(H_2 \mid \theta, R) + L(H_3 \mid \theta, R)$
$\theta_4 \leqq \theta$	acceptance of H_3	$L(H_3 \mid \theta, R)$

It should be noted that at points of discontinuity, $L(\theta, \mid R)$ is defined as the smaller of the two limiting values.

We shall now discuss some monotonicity properties of the function $L(\theta \mid R)$. From the fact that $L(H_{\theta_1} \mid \theta, R_1)$ and $L(H_{\theta_3} \mid \theta, R_2)$ are continuous with continuous first and second derivatives and are monotonically decreasing for all θ with a single point of inflection in the intervals $\theta_1 < \theta < \theta_2$ and $\theta_3 < \theta < \theta_4$ respectively, it follows that

(i) $L(\theta \mid R)$ is monotonically decreasing with negative curvature for $-\infty < \theta \leqq \theta_1$.

(ii) $L(\theta \mid R)$ is monotonically increasing with negative curvature for $\theta_4 \leqq \theta < \infty$.

Making use of (3.3) we have further

(iii) $L(\theta \mid R)$ is monotonically decreasing with negative curvature for $\theta_1 < \theta < \theta_2$.

(iv) $L(\theta \mid R)$ is monotonically increasing with negative curvature for $\theta_3 < \theta < \theta_4$.

(v) For $\theta_2 \leqq \theta \leqq \theta_3$, $\frac{d}{d\theta} L(\theta \mid R) = -\left[\frac{d}{d\theta} L(H_1 \mid \theta, R) + \frac{d}{d\theta} L(H_3 \mid \theta, R)\right]$ is decreasing, since $\frac{d}{d\theta} L(H_1 \mid \theta, R)$ and $\frac{d}{d\theta} L(H_3 \mid \theta, R)$ are increasing. In other words $L(\theta \mid R)$ has negative curvature for $\theta_2 \leqq \theta \leqq \theta_3$.

In the special case when $A = \hat{A} = \frac{1}{B} = \frac{1}{\hat{B}}$ and the origin is taken at $\frac{a_1 + a_2}{2}$ for the sake of convenience, it is easy to see that $L(\theta \mid R)$ is symmetric with respect to the origin and, because of (v), has a local maximum at $\theta = 0$.

5. Choice of the constants A, B, $\hat{A}$, $\hat{B}$ to insure prescribed Lower Bounds for $L(\theta \mid R)$. We shall deal here with the question of choosing A, B, $\hat{A}$ and $\hat{B}$ such that $L(\theta \mid R) \geqq 1 - \gamma_1$ when $\theta \leqq \theta_1$, $L(\theta \mid R) \geqq 1 - \gamma_2$ when $\theta_1 < \theta < \theta_4$, and $L(\theta \mid R) \geqq 1 - \gamma_3$ when $\theta \geqq \theta_4$. From the monotonic properties of the correct decision function it is only necessary to insure that

$$L(\theta_1 \mid R) = 1 - \gamma_1, L(\theta_2 \mid R) = L(\theta_3 \mid R) = 1 - \gamma_2 \text{ and } L(\theta_4 \mid R) = 1 - \gamma_3. \tag{5.1}$$

The following relations will be needed:

$$h_1(\theta_1) = h_2(\theta_3) = 1 = -h_1(\theta_2) = -h_2(\theta_4)$$

$$h_2(\theta_2) = \frac{\theta_3 + \theta_4 - 2\theta_2}{\Delta} = \frac{d - \frac{\Delta}{2}}{\frac{\Delta}{2}} = r \quad \text{(say)}$$

$$h_1(\theta_3) = \frac{\theta_1 + \theta_2 - 2\theta_3}{\Delta} = \frac{-d + \frac{\Delta}{2}}{\frac{\Delta}{2}} = -r$$

where $d = \theta_4 - \theta_2 = \theta_3 - \theta_1 = a_2 - a_1$.

The following four equations are obtained from (5.1):

$$1 - L(H_1 \mid \theta_1, R) = L(H_{\theta_2} \mid \theta_1, R_1) = \frac{1 - B}{A - B} = \gamma_1 \tag{5.2}$$

$$\begin{aligned} 1 - L(H_2 \mid \theta_2, R) &= L(H_1 \mid \theta_2, R) + L(H_3 \mid \theta_2, R) \\ &= \frac{B(A - 1)}{A - B} + \left[\frac{1 - \hat{B}^r}{\hat{A}^r - \hat{B}^r}\right] = \gamma_2 \end{aligned} \tag{5.3}$$

$$\begin{aligned} 1 - L(H_2 \mid \theta_3, R) &= L(H_3 \mid \theta_3, R) + L(H_1 \mid \theta_3, R) \\ &= \frac{1 - \hat{B}}{\hat{A} - \hat{B}} + \left[\frac{B^r(A^r - 1)}{A^r - B^r}\right] = \gamma_2 \end{aligned} \tag{5.4}$$

$$1 - L(H_3 \mid \theta_4, R) = L(H_{\theta_3} \mid \theta_4, R_2) = \frac{\hat{B}(\hat{A} - 1)}{\hat{A} - \hat{B}} = \gamma_3. \tag{5.5}$$

The "bracketed terms" represent quantities less than $\hat{\alpha}$ and β respectively and if r is sufficiently large they can be neglected. This will be made more precise but first let us note the results of neglecting the bracketed terms.

From (5.2) and (5.3) we obtain

$$B(1 - \gamma_1) = \gamma_2, \quad \text{whence} \quad B = \frac{\gamma_2}{1 - \gamma_1}. \tag{5.6}$$

From (5.2) and (5.6)

$$A = \frac{1 - B(1 - \gamma_1)}{\gamma_1} \quad \text{whence} \quad A = \frac{1 - \gamma_2}{\gamma_1}. \tag{5.7}$$

Since the last two equations are obtained from the first two by the permutation $A \to \hat{A}$, $B \to \hat{B}$, $\gamma_1 \to \gamma_2$, $\gamma_2 \to \gamma_3$, we have

$$\hat{B} = \frac{\gamma_3}{1 - \gamma_2}$$

$$\hat{A} = \frac{1 - \gamma_3}{\gamma_2}.$$

If $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$ (say) then $A = \hat{A} = \dfrac{1}{B} = \dfrac{1}{\hat{B}} = \dfrac{1 - \gamma}{\gamma}$.

We shall consider the bracketed quantities negligible if the result of neglecting them produces a change of less than 20% in $[1 - L(\theta \mid R)]$ at $\theta = \theta_2, \theta_3$ respectively, i.e., if

$$\frac{1 - \hat{B}^r}{\hat{A}^r - {}^r\hat{B}} = \frac{1 - \left(\dfrac{\gamma_3}{1 - \gamma_2}\right)^r}{\left(\dfrac{1 - \gamma_3}{\gamma_2}\right)^r - \left(\dfrac{\gamma_3}{1 - \gamma_2}\right)^r} \leqq \frac{\gamma_2}{5} \tag{5.8}$$

and

$$\frac{B^r(A^r - 1)}{A^r - B^r} = \frac{\left(\dfrac{\gamma_2}{1 - \gamma_1}\right)^r \left[\left(\dfrac{1 - \gamma_2}{\gamma_1}\right)^r - 1\right]}{\left(\dfrac{1 - \gamma_2}{\gamma_1}\right)^r - \left(\dfrac{\gamma_2}{1 - \gamma_1}\right)^r} \leqq \frac{\gamma_2}{5}. \tag{5.9}$$

Inequality, (5.9) can be written as

$$\frac{\gamma_2^r[(1 - \gamma_2)^r - \gamma_1^r]}{(1 - \gamma_2)^r(1 - \gamma_1)^r - \gamma_1^r\gamma_2^r} \leqq \frac{\gamma_2}{5}$$

or

$$(1 - \gamma_2)^r\left[\gamma_2^r - \frac{\gamma_2}{5}(1 - \gamma_1)^r\right] \leqq (\gamma_1\gamma_2)^r\left(1 - \frac{\gamma_2}{5}\right).$$

This will certainly hold if

$$\gamma_2^r \leqq \frac{\gamma_2}{5}(1 - \gamma_1)^r$$

or if

$$\left(\frac{\gamma_2}{1 - \gamma_1}\right)^r \leqq \frac{\gamma_2}{5}.$$

Assume that γ_1, γ_2 and γ_3 are each less than $\frac{1}{2}$. Then the last inequality can be written as

$$r \geqq \frac{\log\left(\frac{5}{\gamma_2}\right)}{\log\left(\frac{1-\gamma_1}{\gamma_2}\right)}. \tag{5.10}$$

Starting with (5.8) the same relation is obtained except that γ_1 is replaced by γ_3, namely

$$r \geqq \frac{\log \frac{5}{\gamma_2}}{\log \frac{1-\gamma_3}{\gamma_2}}. \tag{5.11}$$

Let

$$k = \frac{\log \frac{5}{\gamma_2}}{\log \frac{1-\bar{\gamma}}{\gamma_2}}$$

where $\bar{\gamma}$ is the larger of γ_1 and γ_3. Then k is the larger of the right hand members of (5.10) and (5.11). Then for (5.8) and (5.9) to hold it is sufficient that

$$r \geqq k.$$

If $\gamma_2 = .05$ and $0 < \gamma_1, \gamma_3 < .1$ then k is approximately $\frac{2}{1.3} = 1.54$. If $\gamma_2 = .01$ and $0 < \gamma_1, \gamma_3 < .1$ then k is approximately $\frac{2.7}{2} = 1.35$.

We shall now investigate under what conditions the approximate solution obtained above for A, B, $\hat{A}$, $\hat{B}$ are such that acceptance of both θ_1 and θ_4 is impossible.

It follows from (2.1) that the following pair of inequalities are sufficient for the impossibility of accepting both θ_1 and θ_4:

$$\frac{A}{\hat{A}} = \frac{\gamma_2}{\gamma_1}\frac{1-\gamma_2}{1-\gamma_3} \leqq 1; \qquad \frac{B}{\hat{B}} = \frac{\gamma_2}{\gamma_3}\frac{1-\gamma_2}{1-\gamma_1} \leqq 1. \tag{5.12}$$

If $\gamma_1 \neq \gamma_3$ let the smaller and larger of the pair (γ_1, γ_3) be denoted by $\underline{\gamma}$ and $\bar{\gamma}$ respectively. Since $1 - \underline{\gamma} > 1 - \bar{\gamma}$, then

$$\frac{\gamma_2(1-\gamma_2)}{\bar{\gamma}(1-\underline{\gamma})} < \frac{\gamma_2(1-\gamma_2)}{\underline{\gamma}(1-\bar{\gamma})}$$

and we need only consider one of the two inequalities in (5.12). The condition $\gamma_2 < \underline{\gamma}$ will in general satisfy (5.12). More precisely if all the γ's are restricted to the interval (0, .1) then

$$\frac{9}{10} \leqq \frac{1-\gamma_2}{1-\underline{\gamma}} < \frac{1-\gamma_2}{1-\bar{\gamma}} \leqq \frac{10}{9}$$

and it is sufficient for the validity of (5.12) that $\gamma_2 \leqq (.9)\,\underline{\gamma}$.

If $\gamma_1 = \gamma_3 = \gamma$ (say) then the two inequalities reduce to one

$$\gamma_2^2 - \gamma_2 + \gamma - \gamma^2 \geqq 0$$

which can be written as

$$(\gamma_2 - \gamma)(\gamma_2 - 1 + \gamma) \geqq 0.$$

Since the inequality $\gamma_2 \geqq 1 - \gamma$ is impossible when all γ's are $< \frac{1}{2}$, we see that $\gamma_2 \leq \gamma$ is sufficient for the validity of (5.12) when $\gamma_1 = \gamma_3 = \gamma < \frac{1}{2}$.

There remains the problem of finding an approximate solution for equations (5.2) to (5.5) when $r < k$. Since

$$r = \frac{d - \frac{\Delta}{2}}{\frac{\Delta}{2}} = \frac{\theta_3 - \theta_2 + \frac{\Delta}{2}}{\frac{\Delta}{2}} \geqq 1$$

we merely have to consider the interval $1 \leqq r < k$.

The following approximations are used

$$(5.13)\qquad \begin{array}{lll} \dfrac{1 - B}{A - B} \sim \dfrac{1}{A}; & \dfrac{B(A - 1)}{A - B} \sim B; & \dfrac{1 - \hat{B}^r}{\hat{A}^r - \hat{B}^r} \sim \dfrac{1}{\hat{A}^r} \\ \dfrac{1 - \hat{B}}{\hat{A} - \hat{B}} \sim \dfrac{1}{\hat{A}}; & \dfrac{B^r(A^r - 1)}{A^r - B^r} \sim B^r; & \dfrac{\hat{B}(\hat{A} - 1)}{\hat{A} - \hat{B}} \sim \hat{B}, \end{array}$$

which upon substitution yield

$$(5.14)\qquad A = \frac{1}{\gamma_1}$$

$$(5.15)\qquad \hat{B} = \gamma_3$$

$$(5.16)\qquad B + \frac{1}{\hat{A}^r} = \gamma_2$$

$$(5.17)\qquad \frac{1}{\hat{A}} + B^r = \gamma_2 .$$

Subtraction of (5.17) from (5.16) shows that $B = \frac{1}{\hat{A}}$ is a solution. Substituting this result back in (5.16) leads to the equation

$$(5.18)\qquad B + B^r = \gamma_2 .$$

It can easily be verified that between zero and unity this equation has exactly one root. Since $1 \leqq r < \infty$, the root of the above equation lies between $\frac{\gamma_2}{2}$ and γ_2.

Taking γ_2 as a first approximation for B and substituting $\gamma_2 + \epsilon$ for B in (5.18), we obtain

$$\epsilon + (\gamma_2 + \epsilon)^r = 0.$$

Expanding $(\gamma_2 + \epsilon)^r$ in a power series in ϵ and neglecting second and higher order terms, the above equation gives

$$\epsilon \sim \frac{\gamma_2^r}{1 + r\gamma_2^{r-1}}.$$

Thus,

$$B = \frac{1}{A} \sim \gamma_2 - \frac{\gamma_2^r}{1 + r\gamma_2^{r-1}} = \frac{\gamma_2[1 + (r-1)\gamma_2^{r-1}]}{1 + r\gamma_2^{r-1}}. \tag{5.19}$$

It is necessary to investigate under what conditions the above approximate solution satisfies (5.2) to (5.5) to within a 20% error in $[1 - L(\theta/R)]$, i.e., such that

$$-\frac{\gamma_1}{5} < \frac{\gamma_1(1-B)}{1-\gamma_1 B} - \gamma_1 < \frac{\gamma_1}{5} \tag{5.20}$$

$$-\frac{\gamma_3}{5} < \frac{\gamma_3(1-B)}{1-\gamma_3 B} - \gamma_3 < \frac{\gamma_3}{5} \tag{5.21}$$

$$-\frac{\gamma_2}{5} < \frac{B(1-\gamma_1)}{1-\gamma_1 B} + \frac{B^r(1-\gamma_3^r)}{1-(\gamma_3 B)^r} - \gamma_2 < \frac{\gamma_2}{5} \tag{5.22}$$

$$-\frac{\gamma_2}{5} < \frac{B(1-\gamma_3)}{1-\gamma_3 B} + \frac{B^r(1-\gamma_1^r)}{1-(\gamma_1 B)^r} - \gamma_2 < \frac{\gamma_2}{5} \tag{5.23}$$

where for B the value in (5.19) is understood.

It can be shown that if γ_1, γ_2, γ_3, are each between zero and .1 then the inequalities (5.20) to (5.23) hold. Furthermore it can be shown that if, in addition $\gamma_2 \leqq \min(\gamma_1, \gamma_3)$ then also the inequalities (2.1) hold. The latter inequalities are sufficient to ensure the impossibility of accepting both θ_1 and θ_4.

6. Bounds for the ASN Function. First we shall derive lower bounds for the ASN function. Let $E(n/\theta, R)$ denote the expected value of n when θ is the true mean and R is the sequential rule employed. For $\theta < \theta_2$ the probability of coming to a decision first with R_2 is large and therefore

$$E(n/\theta, R) \sim E(n/\theta, R_1) \; \theta < \theta_2.$$

From the definition of R it follows that

$$E(n/\theta, R) > E(n/\theta, R_1) \qquad \text{for all } \theta.$$

Hence $E(n/\theta, R_1)$ serves as a close lower bound when $\theta < \theta_2$.

Similarly

$$E(n/\theta, R) \sim E(n/\theta, R_2) \qquad \text{for } \theta > \theta_3$$

$$E(n/\theta, R) > E(n/\theta, R_2) \qquad \text{for all } \theta.$$

Hence $E(n/\theta, R_2)$ serves as a close lower bound for $\theta > \theta_3$.

Combining the above we have

$$E(n/\theta, R) > \text{Max}\,[E(n/\theta, R_1)\;,\; E(n/\theta, R_2)] \tag{6.1}$$

where, neglecting the excess over the boundary,

$$E(n/\theta, R_1) = \frac{L(H_{\theta_1}/\theta, R_1) \log B + L(H_{\theta_2}/\theta, R_1) \log A}{\frac{\Delta}{\sigma^2}(\theta - a_1)} \tag{6.2}$$

$$E(n/\theta, R_2) = \frac{L(H_{\theta_3}/\theta, R_2) \log \hat{B} + L(H_{\theta_4}/\theta, R_2) \log \hat{A}}{\frac{\Delta}{\sigma_2}(\theta - a_2)} \tag{6.3}$$

Formula (6.1) gives a valid lower bound over the whole range of θ, but this lower bound will not be very close in the interval (θ_2, θ_3), particularly in the neighbourhood of the mid-point $\frac{\theta_2 + \theta_3}{2}$. The authors were not able to find any simple method for obtaining a closer lower bound in this interval. The upper bound given later in this section will, however, be fairly close also in the interval (θ_2, θ_3) and can be used as an approximation to the exact value.

We shall now derive upper bounds for the ASN function. Let R_1^* be the following rule: "Continue to take observations until R_1 accepts θ_1." Since this implies the rejection of θ_4 at the same or at a previous stage, it follows that R must terminate not later than R_1^*. Hence

$$E(n/\theta, R_1^*) \geqq E(n/\theta, R). \tag{6.4}$$

As a matter of fact one can easily verify that $E(n/\theta, R_1^*) > E(n/\theta, R)$. Thus $E(n/\theta, R_1^*)$ is an upper bound for $E(n/\theta, R)$. This upper bound will be close when the probability of accepting θ_1 is high, i.e., for $\theta \leqq \theta_1$.

By the general formula

$$E(n) = \frac{E\left(\sum_{i=1}^{n} z_i\right)}{E(z)}$$

(see p. 53 [1]) we obtain, upon neglecting the excess over the boundary,

$$E(n/\theta, R_1^*) = \frac{\log B}{\frac{\Delta}{\sigma^2}(\theta - a_1)}. \tag{6.5}$$

This coincides with (6.2) when $L(H_{\theta_2}/\theta, R_1) = 0$.

Similarly, if R_2^* denotes the rule of continuing until R_2 accepts θ_4, then

$$E(n/\theta, R_2^*) > E(n/\theta, R) \tag{6.6}$$

$$E(n/\theta, R_2^*) = \frac{\log \hat{A}}{\frac{\Delta}{\sigma^2}(\theta - a_2)} \tag{6.7}$$

and this will be a close upper bound for $\theta \geqq \theta_4$.

If $A = \hat{A} = \frac{1}{B} = \frac{1}{\hat{B}}$ and if $a_1 + a_2 = 0$ the above results reduce to

$$(6.8) \qquad E(n/\theta, R) \gtrsim E(n/\theta, R_1^*) = \frac{-h}{\lambda + \theta} \qquad \text{for} \qquad \theta \leqq \theta_1$$

$$(6.9) \qquad E(n/\theta, R) \gtrsim E(n/\theta, R_2^*) = \frac{h}{\theta - \lambda} \qquad \text{for} \qquad \theta \geqq \theta_4$$

where the symbol $\gtrsim$ stands for a close inequality, and where

$$h = \frac{\sigma^2}{\Delta} \log A \qquad \text{and} \qquad \lambda = a_2 = -a_1 .$$

To establish an upper bound for the ASN function in the interval $\theta_2 < \theta < \theta_3$ we shall restrict ourself to the case where $A = \hat{A} = \frac{1}{B} = \frac{1}{\hat{B}}$. These relations are fulfilled by the approximate values of A, B, $\hat{A}$, $\hat{B}$ suggested in section 5 when $\gamma_1 = \gamma_2 = \gamma_3$ and $r \geqq k$. We shall choose the origin to be at $\frac{a_1 + a_2}{2}$, i.e., we put $\frac{a_1 + a_2}{2} = 0$. Then the vertex P of the triangle (P_1, P_2, P) in diagram 1 lies on the abscissa axis and $OP_1 = OP_2 = h$. The abscissa of the vertex P is $\frac{h}{\lambda} = N$ (say) where $\lambda = a_2 = -a_1$. Let $y = \sum_{i=1}^{N} X_i$ represent the sum of the first N observations. Let R_{23} denote the rule: "Continue until both θ_2 and θ_3 are accepted". This is tantamount to neglecting the two outer lines in diagram 1, i.e., the acceptance lines for θ_1 and θ_4. Then clearly,

$$(6.10) \qquad E(n/\theta, R_{23}) > E(n/\theta, R).$$

When θ lies between θ_2 and θ_3 this inequality will be close, since the probability of crossing either of the two outer lines is then small.

However $E(n/\theta, R_{23})$ was found difficult to compute and it was necessary to consider instead the rule R'_{23}: "Take N observations. If $y = \sum_{i=1}^{N} X_i < 0$ then continue until θ_2 is accepted. If $y > 0$ then continue until θ_3 is accepted".[3] Clearly,

$$(6.11) \qquad E(n/\theta, R'_{23}) > E(n/\theta, R_{23}).$$

This inequality, however, will be close only if the probability of concluding the test before N observations, given that $\theta_2 < \theta < \theta_3$, is small.

Some investigations by the authors seem to indicate that the inequality (6.11) will be close when $\Delta < \lambda$. This inequality is likely to be fulfilled in practical problems.

We shall now proceed to determine the value of $E(n/\theta, R'_{23})$. Neglecting the excess over the boundary, we have

$$(6.12) \qquad E\left(n/\theta, R'_{23}, \sum_{i=1}^{N} x_i = y\right) = \frac{h}{\lambda} + \frac{y}{\lambda - \theta} \qquad \text{for} \qquad y > 0$$

[3] The event $y = 0$ has probability zero and it is indifferent what rule is adopted for that case.

and

$$(6.13) \qquad E\left(n/\theta, R'_{23}, \sum_{i=1}^{N} x_i = y\right) = \frac{h}{\lambda} - \frac{y}{\lambda + \theta} \qquad \text{for} \qquad y < 0$$

where, for any condition C, $E(n/\theta, R, C)$ denotes the conditional expected value of n given that the true mean is θ, that R is the sequential rule used and that the condition C is fulfilled.

Multiplying with the density of y and then integrating with respect to y, we obtain after simplification

$$(6.14) \quad E(n/\theta, R'_{23}) = \frac{1}{\lambda^2 - \theta^2}\left[h\lambda + 2h\theta\phi\left(\frac{\theta}{\sigma}\sqrt{\frac{h}{\lambda}}\right) + 2\sigma\sqrt{\frac{h\lambda}{2\pi}}\, e^{-(h\theta^2/2\lambda\sigma^2)}\right]$$

where $\phi(x) = \int_0^x \frac{e^{-(y^2/2)}}{\sqrt{2\pi}}\, dy$, and $\theta_2 < \theta < \theta_3$.

In particular, for $\theta = 0$ we get

$$(6.15) \qquad E(n/\theta = 0, R'_{23}) = \frac{h}{\lambda} + \frac{\sigma}{\lambda^2}\sqrt{\frac{2h\lambda}{\pi}}.$$

To establish a close upper bound for $\theta_3 < \theta < \theta_4$ we must bring the line of acceptance of θ_4 into account. The line of acceptance of θ_1 can be disregarded since the probability of accepting θ_1 is very small.

We therefore define the rule R_{34} as follows:

"Continue with R_1 until θ_2 is accepted and with R_2 until either θ_3 or θ_4 is accepted."

Since the ASN function for R_{34} is difficult to compute we define a modified rule R'_{34} as follows:

"Proceed to take $N\left(=\frac{h}{\lambda}\right)$ observations without regard to any rule. If $y = \sum_{i=1}^{N} X_i < 0$ then continue only with R_1 until θ_2 is accepted. If $0 < y < 2h$ then continue only with R_2 until either θ_3 or θ_4 is accepted. If $y \geqq 2h$ then stop taking observations and accept H_3."

It is clear that the following inequalities hold

$$(6.16) \qquad E(n/\theta, R'_{34}) > E(n/\theta, R_{34}) > E(n/\theta, R).$$

The proximity of $E(n/\theta, R_{34})$ and $E(n/\theta, R)$, as stated above, is based on the fact that the probability of accepting θ_1, when $\theta_3 < \theta < \theta_4$, is small.

The proximity of $E(n/\theta, R_{34})$ and $E(n/\theta, R'_{34})$ is assured if the probability of terminating with R_{34} (and with R) before N observations is small. It can be shown that the latter condition is fulfilled when $\Delta < \lambda$. In terms of the quantity r defined in Section 5 this can be written as $r > 3$.

To determine the value of $E(n/\theta, R'_{34})$ the following two preliminary results will be needed:

If $0 < y < 2h$,

$$(6.17)\quad E\left(n/\theta, R'_{34}, \sum_{i=1}^{N} x_i = y\right) = \frac{h}{\lambda} + \frac{2h - y - 2h\left[\dfrac{1 - e^{-(2/\sigma^2)(\lambda-\theta)(2h-y)}}{1 - e^{-(4h/\sigma^2)(\lambda-\theta)}}\right]}{\theta - \lambda} = C \text{ (say)}.$$

If $y < 0$,

$$(6.18)\quad E\left(n/\theta, R'_{34}, \sum_{i=1}^{N} x_i = y\right) = \frac{h}{\lambda} - \frac{y}{\lambda + \theta} = D \text{ (say)}.$$

Both are easily obtained from formula (7.25) on p. 123 of [1].

Multiplying with the density of y and integrating with respect to y, we obtain after simplification

$$(6.19)\quad \begin{aligned} E(n/\theta, R'_{34}) = \frac{h}{\lambda} &+ \left[\phi\left(\frac{2\lambda - \theta}{\sigma}\sqrt{\frac{h}{\lambda}}\right) + \phi\left(\frac{\theta}{\sigma}\sqrt{\frac{h}{\lambda}}\right)\right] \\ &\cdot \frac{h}{(\lambda - \theta)}\left(\frac{\theta}{\lambda} - \frac{2e^{-(2h(\lambda-\theta)/\sigma^2)}}{1 + e^{-2h(\lambda-\theta)/\sigma^2}}\right) \\ &+ \frac{\sigma}{\lambda(\lambda - \theta)}\sqrt{\frac{h\lambda}{2\pi}}\left[e^{-(h\theta^2/2\lambda\sigma^2)} - e^{-h(2\lambda-\theta)^2/2\lambda\sigma^2}\right] \\ &- \frac{h\theta}{2\lambda(\lambda + \theta)}\left[1 - 2\phi\left(\frac{\theta}{\sigma}\sqrt{\frac{h}{\lambda}}\right)\right] + \frac{\sigma}{\lambda(\lambda + \theta)}\sqrt{\frac{h\lambda}{2\pi}}\, e^{-(h\theta^2/2\lambda\sigma^2)}. \end{aligned}$$

Formula (6.19) is an improvement on (6.14) as it will give for any θ a smaller upper bound, but in the neighborhood of the origin the difference is insignificant.

For $\theta = \lambda$ we obtain from (6.19) using L'Hopital's rule

$$(6.20)\quad \begin{aligned} E(n/\lambda, R'_{34}) = \frac{h^2}{\sigma^2} &- \frac{h}{4\lambda\sigma^2}(4h\lambda - 3\sigma^2) \\ &\cdot \left[1 - 2\phi\left(\frac{\sqrt{h\lambda}}{\sigma}\right)\right] + \left(\frac{h\lambda + \sigma^2}{2\lambda^2\sigma}\right)\sqrt{\frac{h\lambda}{2\pi}}\, e^{-(h\lambda/2\sigma^2)}. \end{aligned}$$

If $\dfrac{\sqrt{h\lambda}}{\sigma} > 2.5$, the above formula can be approximated by

$$(6.21)\quad E(n/\lambda, R'_{34}) \sim \frac{h^2}{\sigma^2} + \frac{2\sigma}{\lambda^2}\sqrt{\frac{h\lambda}{2\pi}}\, e^{-(h\lambda/2\sigma^2)}.$$

Since the right hand member above lies between $\dfrac{h^2}{\sigma^2}$ and $(1.002)\dfrac{h^2}{\sigma^2}$ when $\dfrac{\sqrt{h\lambda}}{\sigma} > 2.5$ then for practical purposes

$$(6.22)\quad E(n/\lambda, R'_{34}) \sim \frac{h^2}{\sigma^2} \qquad \text{when } \left(\frac{\sqrt{h\lambda}}{\sigma} > 2.5\right).$$

An upper bound for $E(n/\theta, R)$ for $\theta_1 < \theta < \theta_2$ can be obtained by defining R_{12} and R'_{12} in an analagous way to R_{34} and R'_{34}. Because of reasons of symmetry, $E(n/\theta, R'_{12})$ can be obtained from (6.19) by replacing θ by $-\theta$.

The method used for obtaining upper bounds for $E(n/\theta, R)$ can easily be extended to the more general case when the equalities $A = \hat{A} = \frac{1}{B} = \frac{1}{\hat{B}}$ do not necessarily hold. However, the resulting formulas are more cumbersome and we shall merely give without proof the upper bound corresponding to (6.14). This upper bound becomes

$$E(n/\theta, R'_{23}) = N + \left(\frac{N\theta - h_3}{\lambda - \theta}\right)\left[\frac{1}{2} - \phi(a)\right] + \left(\frac{h_3 - N\theta}{\lambda + \theta}\right)\left[\frac{1}{2} - \phi(b)\right] + \sigma\sqrt{\frac{N}{2\pi}}\left[\frac{e^{-a^2/2}}{\lambda - \theta} + \frac{e^{-b^2/2}}{\lambda + \theta}\right]$$

where

$$h_{11} = \frac{\sigma^2}{\Delta}\log A \qquad h_{10} = \frac{\sigma^2}{\Delta}\log B$$

$$h_{21} = \frac{\sigma^2}{\Delta}\log \hat{A} \qquad h_{20} = \frac{\sigma^2}{\Delta}\log \hat{B}$$

$$a_2 = -a_1 = \lambda$$

$$N = \frac{h_{11} - h_{20}}{2\lambda}; \qquad a = \frac{h_3 - N\theta}{\sigma\sqrt{N}}; \qquad b = \frac{h_3 + N\theta}{\sigma\sqrt{N}}; \qquad h_3 = \frac{h_{11} + h_{20}}{2}.$$

7. An Example. We shall consider the following example

$\sigma^2 = 1$, $\theta_1 = -\frac{5}{16}$, $\theta_2 = -\frac{3}{16}$, $\theta_3 = \frac{3}{16}$, $\theta_4 = \frac{5}{16}$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma = .029$

then

$$A = \hat{A} = \frac{1}{B} = \frac{1}{\hat{B}} = \frac{1 - \gamma}{\gamma} = 33.5 \qquad r = 7 \gg 3 > k \sim 1.47$$

and

$$h = \frac{\sigma^2}{\Delta}\log A = 28, \lambda = \frac{\theta_3 + \theta_4}{2} = \tfrac{1}{4}, \Delta = \theta_2 - \theta_1 = \theta_4 - \theta_3 = \tfrac{1}{8}.$$

Using formulas (6.1) and (6.7) the following upper and lower bounds were obtained

θ	$\frac{5}{16}$	$\frac{6}{16}$	$\frac{7}{16}$	$\frac{8}{16}$	$\frac{9}{16}$	$\frac{10}{16}$	$\frac{12}{16}$	$\frac{14}{16}$	$\frac{16}{16}$	$\frac{18}{16}$	$\frac{20}{16}$
Upper bound...........	448	224	149	112	89.6	74.7	56	44.8	37.3	32	28
Lower bound...........	421	224	149	112	89.6	74.7	56	44.8	37.3	32	28

Formulas (6.14) and (6.1) yield

θ	0	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$
Upper Bound.........	146	163	229	450
Lower Bound..........	112	149	224	421

In the neighborhood of the origin the true value is very nearly the upper bound. From formulas (6.19), (6.22) and (6.1) we obtain

θ	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{5}{16}$
Upper Bound..........	422	784.5	423
Lower Bound..........	421	784	421

As shown above for the end points of the indifference zone, (6.19) gives better results than (6.14) or (6.7). This is as it should be since (6.19) takes into account possibilities omitted in (6.14) and (6.7). The greater accuracy of (6.19) is offset by a slight increase in computation.

In the graph of the Bounds of the ASN function shown in Figure 2, a single curve is shown wherever the upper and lower bound are sufficiently close to each other.

Since (6.14) contains an even function of θ and since elsewhere the corresponding bounds are mirror images with respect to $\theta = 0$, the bounds for negative θ are exactly the same as those for the corresponding positive θ.

Consider the following non-sequential rule applied to our problem. With a fixed number N_0 of observations compute the mean $\bar{x}$ and accept H_1 if $\bar{x}$ falls in the interval $(-\infty, a_1)$, accept H_2 if $\bar{x}$ falls in $[a_1, a_2]$ and accept H_3 if $\bar{x}$ falls in (a_2, ∞). This is certainly a reasonable procedure. One can also verify that no other non-sequential rule exists that is uniformly better (for all possible values of θ) than the one under consideration.

The two decision procedures become comparable if we introduce the indifference zones and define a wrong decision in the non-sequential case exactly as was done for our sequential procedure (see Section 1).

For the non-sequential case (just as in the sequential case) the probability of a wrong decision will be discontinuous at θ_1, θ_2, θ_3 and θ_4. At each of these points there will be a left-sided and right-sided limit, different from each other. As in the sequential case we shall take the probability of a wrong decision at a discontinuity point to be equal to the larger of the left and right hand limits. One can easily verify that the maximum probability of a wrong decision occurs at $\theta = \theta_3$ (which is equal to the value at $\theta = \theta_2$).

We then determine N_0 by setting the maximum probability of a wrong decision equal to γ, i.e.

$$\phi\left(\frac{d - \Delta/2}{\sigma} \sqrt{N_0}\right) + \phi\left(\frac{\Delta}{2\sigma} \sqrt{N_0}\right) = 1 - \gamma. \tag{7.1}$$

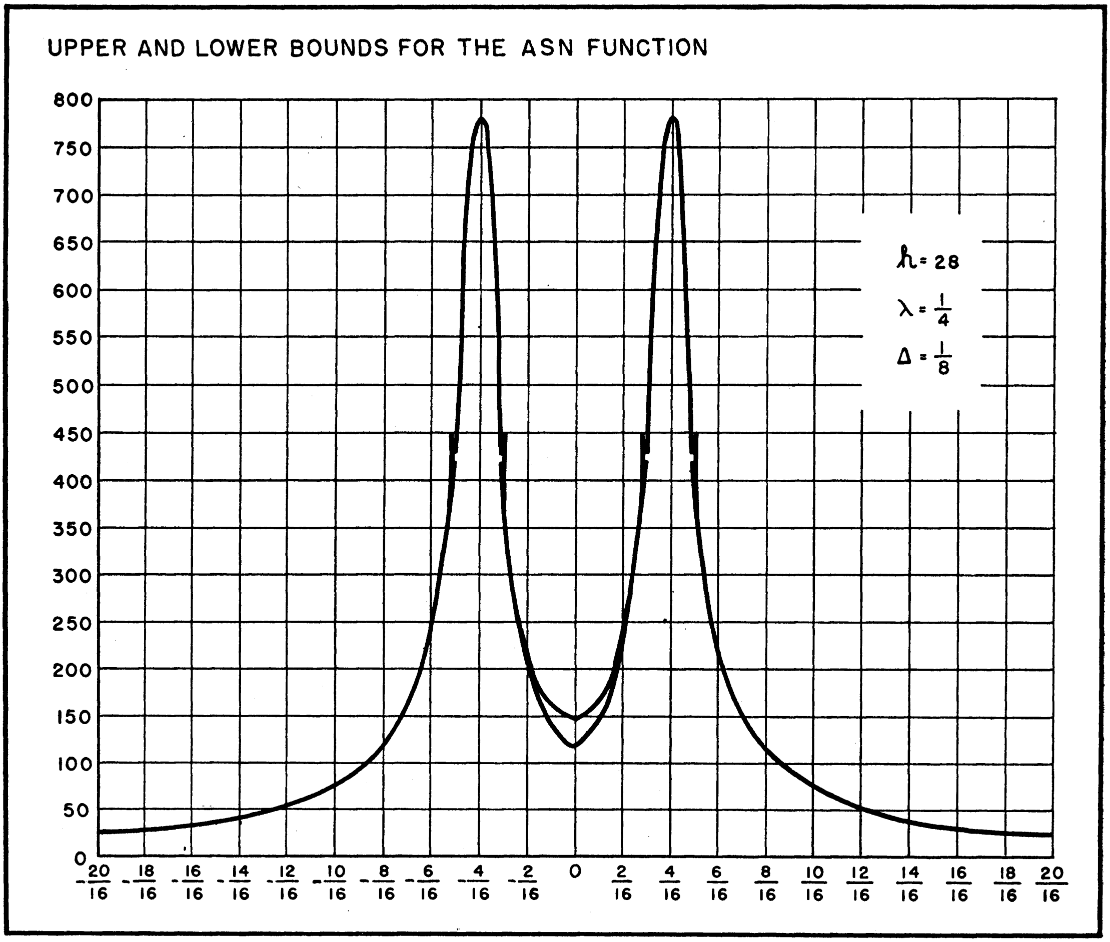

FIGURE 2

For the particular problem considered above, this gives $N_0 = 915.4$. Hence 916 observations are required in order to ensure that this non-sequential procedure will have the maximum probability $\gamma = .029$ of a wrong decision. This is to be compared with the maximum over all θ of the ASN function in the sequential procedure, which was 784.5.

Returning to (7.1) we shall derive lower and upper bounds for the root of that equation. Since

$$\infty > \frac{d - \Delta/2}{\sigma} \sqrt{N_0} \geqq \frac{\Delta}{2\sigma} \sqrt{N_0}$$

it is clear that the root of the equation

$$\phi\left(\frac{\Delta}{2\sigma}\sqrt{N_0}\right) + \phi\left(\frac{\Delta}{2\sigma}\sqrt{N_0}\right) = 1 - \gamma$$

is an upper bound for the root of (7.1) and that the root of the equation

$$\phi(\infty) + \phi\left(\frac{\Delta}{2\sigma}\sqrt{N_0}\right) = 1 - \gamma$$

or

$$\phi\left(\frac{\Delta}{2\sigma}\sqrt{N_0}\right) = \frac{1}{2} - \gamma$$

is a lower bound for the root of (7.1). We shall compare the value of $x = \frac{\Delta}{2\sigma}\sqrt{N_0}$ with the value of $y = \frac{\Delta}{2\sigma}\sqrt{\underset{\theta}{\text{Max}}\ \text{ASN}}$. Since

$$\underset{\theta}{\text{Max}}\ (\text{ASN function}) \sim \frac{h^2}{\sigma^2} = \frac{\sigma^2}{\Delta^2}\left(\log\frac{1-\gamma}{\gamma}\right)^2 \text{ (for sufficiently small } \frac{\Delta}{d}\text{)}.$$

then

$$y = \frac{\Delta}{2\sigma}\sqrt{\underset{\theta}{\text{Max}}\ \text{ASN}} \sim \frac{1}{2}\log\frac{1-\gamma}{\gamma} \text{ (for sufficiently small } \frac{\Delta}{d}\text{)}.$$

The following table gives upper and lower bounds for x and the corresponding value of y for the type of example under consideration, i.e., when $A = \hat{A} = \frac{1}{B} \doteq \frac{1}{\hat{B}}$ and $r \geqq k$.

γ	.001	.002	.005	.008	.01	.05	.1
$\underline{x}$ and $\bar{x}$	3.08–3.31	2.87–3.10	2.57–2.81	2.41–2.65	2.33–2.58	1.64–1.96	1.28–1.65
y	3.45	3.11	2.65	2.41	2.30	1.47	1.10

As the table shows[4] for $.1 > \gamma > .008$

$$x > \underline{x} > y$$

[4] Actually, the inequality in question is shown only for the values of γ given in the table. However it can be verified that the inequality remains valid for all values of γ between .1 and .008.

and hence

$$N_0 > \underset{\theta}{\text{Max}}\ \text{ASN} \qquad \left(\text{for sufficiently small } \frac{\Delta}{d}\right).$$

The statement and the table above are not meant to delimit the region in which the sequential rule is superior to the non-sequential procedure.

REFERENCES

[1] Abraham Wald, *Sequential Analysis*, John Wiley and Sons, 1947.

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