THE IMPOSSIBILITY OF CERTAIN SYMMETRICAL BALANCED INCOMPLETE BLOCK DESIGNS

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Introduction and Summary. An arrangement of v varieties or treatments in b blocks of size k, $(k < v)$, is known as a balanced incomplete block design if every variety occurs in r blocks and any two varieties occur together in λ blocks. These parameters obviously satisfy the equations

(1) $$bk = vr$$

(2) $$\lambda(v - 1) = r(k - 1).$$

Fisher [1] has also proved that the inequality

(3) $$b \geq v, \qquad r \geq k$$

must hold. If v, b, r, k and λ are positive integers satisfying (1), (2) and (3), then a balanced incomplete block design with these parameters possibly exists, but the actual existence of a combinatorial solution is not ensured. These conditions are thus necessary but not sufficient for the existence of a design. Fisher and Yates in their tables [2] have listed all designs with $r \leq 10$ and given combinatorial solutions, where known. A balanced incomplete block design in which $b = v$, and hence $r = k$ is called a symmetrical balanced incomplete block design. The impossibility of the symmetrical designs with parameters $v = b = 22$, $r = k = 7$, $\lambda = 2$ and $v = b = 29$, $r = k = 8$, $\lambda = 2$ was first demonstrated by Hussain [3], [4] essentially by the method of enumeration. The object of the present note is to give an alternative simple proof of the impossibility of these designs and to show that the only unknown remaining symmetrical design in Fisher and Yates' tables, *viz.* $v = b = 46$, $r = k = 10$, $\lambda = 2$, is definitely impossible. Symmetrical designs with $\lambda \leq 5$, $r, k \leq 20$, which are impossible combinatorially, are also listed.

1. A necessary condition for the existence of a symmetrical balanced incomplete block design when v is even.

Theorem 1. *A necessary condition for the existence of a symmetrical balanced incomplete block design with parameters v, r and λ, where v is even, is that $r - \lambda$ be a perfect square.*

Proof. Let $N = (n_{ij})$ be a square matrix of v rows and v columns where

(4) $$n_{ij} = 1 \text{ or } 0$$

according as the i-th treatment does or does not occur in the j-th block. Put

(5) $$B = NN'$$

Since every treatment occurs in r blocks and every pair of treatments in λ blocks, we have, if the design is possible,

$$B = \begin{pmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \cdots & & \cdots & \cdot \\ \lambda & \lambda & \cdots & r \end{pmatrix}. \tag{6}$$

Subtracting the first column from all the other columns and then adding to the first row all the other rows, we see that

$$\begin{aligned} |B| &= [r + \lambda(v - 1)](r - \lambda)^{v-1} \\ &= r^2(r - \lambda)^{v-1} \text{ from (2).} \end{aligned} \tag{7}$$

But from (5)

$$|B| = |N|^2.$$

Since $|N|$ is integral, it follows that $(r - \lambda)^{v-1}$ is the square of an integer, and hence if v is even, $r - \lambda$ must be a perfect square.

COROLLARY. *The following symmetrical designs are impossible.*

(A_1)	$v = b = 22$	$r = k = 7$	$\lambda = 2$
(A_2)	$v = b = 46$	$r = k = 10$	$\lambda = 2$
(A_3)	$v = b = 92$	$r = k = 14$	$\lambda = 2$
(A_4)	$v = b = 106$	$r = k = 15$	$\lambda = 2$
(A_5)	$v = b = 172$	$r = k = 19$	$\lambda = 2$
(A_6)	$v = b = 34$	$r = k = 12$	$\lambda = 4.$

As already mentioned in the introduction, the impossibility of (A_1) has been proved by Hussain [3], but for the design (A_2) it was hitherto unknown whether or not a solution is possible and it was left as a blank in the latest edition of Fisher and Yates' tables.

2. Application of method of Bruck and Ryser.

In a recent paper Bruck and Ryser [5] have proved the impossibility of some finite projective planes with the help of the properties of matrices whose elements are integers. Their method is immediately applicable to our own problem.

Let A and B be two symmetric matrices of order n with elements in the rational field. The matrices A and B are congruent, written $A \sim B$, provided there exists a nonsingular matric C with elements in the rational field, such that $A = C'BC$. The congruence of matrices satisfies the usual requirements of an "equals" relationship.

If A is an integral symmetric matrix of order n and rank n, we can always construct an integral diagonal matrix $D = (d_1, \cdots, d_n)$, where $d_i \neq 0$, $i = 1, 2, \cdots n$ such that $D \sim A$. The number of negative terms i, called the index of A, is an invariant by Sylvester's Law.

Define $d = (-1)^i\delta$ where δ is the square-free positive part of $|A|$. Then since $|B| = |C|^2|A|$, d is another invariant of A.

Now let A be a nonsingular and symmetric integral matrix of order n. Let D_r be the leading principal minor determinant of order r and suppose that $D_r \neq 0$ for $r = 1, 2, \cdots n$. Define

$$C_p(A) = (-1, -D_n)_p \prod_{j=1}^{n-1} (D_j, -D_{j+1})_p \tag{9}$$

for every odd prime p where $(m, m')_p$ is the Hilbert norm-residue symbol for arbitrary non-zero integers m and m' and for every prime p. The following two theorems are given in the collected works of Hilbert [6].

THEOREM (A). *If m and m' are integers not divisible by the odd prime p, then*

$$(m, m')_p = +1 \tag{10}$$

$$(m, p)_p = (p, m)_p = (m/p), \tag{11}$$

where (m/p) is the Legendre symbol. Moreover, if $m \equiv m' \not\equiv 0 \bmod p$, then

$$(m, p)_p = (m', p)_p . \tag{12}$$

THEOREM (B). *For arbitrary non-zero integers m, m', n, n' and for every prime p,*

$$(-m, m)_p = +1 \tag{13}$$

$$(m, n)_p = (n, m)_p \tag{14}$$

$$(mm', n)_p = (m, n)_p(m', n)_p \tag{15}$$

$$(m, nn')_p = (m, n)_p(m, n')_p . \tag{16}$$

From the above it is easy to prove that for p an odd prime and every positive integer m,

$$(m, m + 1)_p = (-1, m + 1)_p \tag{17}$$

$$\prod_{j=1}^{m} (j, j + 1)_p = ((m + 1)!, - 1)_p . \tag{18}$$

We can now state the fundamental Minkowski-Hasse Theorem [7].

THEOREM (C). *Let A and B be two integral symmetric matrices of order n and rank n. Suppose further that the leading principal minor determinants of A and B are different from zero. Then $A \sim B$ if and only if A and B have the same invariants i, d and C_p for every odd prime p.*

3. A necessary condition for the existence of a symmetrical balanced incomplete block design for any integer v.

Suppose the symmetrical design with parameters v, r and λ exists. Then with the previous definition of N and B,

$$B = NN' = \begin{pmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \cdots & & \cdots & \\ \lambda & \lambda & \cdots & r \end{pmatrix}.$$

Subtracting the last row from the remaining rows and then subtracting the last column from all the other columns, we get

$$(19)\qquad Q = \begin{bmatrix} 2(r-\lambda) & (r-\lambda) & \cdots & (r-\lambda) & -(r-\lambda) \\ (r-\lambda) & 2(r-\lambda) & \cdots & (r-\lambda) & -(r-\lambda) \\ \cdots & \cdots & & \cdots & \\ (r-\lambda) & (r-\lambda) & \cdots & 2(r-\lambda) & -(r-\lambda) \\ -(r-\lambda) & -(r-\lambda) & \cdots & -(r-\lambda) & r \end{bmatrix}.$$

Obviously $Q \sim B$. But $B \sim I$. Hence $Q \sim I$ and, therefore, since Q and I satisfy all the conditions of Theorem C, they must have the same invariants i, d and C_p.

Let D_j denote the leading principal minor determinant of Q of order j. Then

$$(20)\qquad D_j = (r-\lambda)^j(j+1) \quad \text{for } j = 1, 2, \cdots, v-1$$

$$(21)\ \text{and}\qquad D_v = |B| = r^2(r-\lambda)^{v-1}.$$

Then, omitting p for convenience,

$$C_p(Q) = (-1, -D_v)(D_{v-1}, -D_v)\prod_{j=1}^{v-2}(D_j, -D_{j+1}).$$

We use (10) $\cdots$, (18) in deriving the value of $C_p(Q)$.
Now

$$\begin{aligned}(-1, -D_v)&(D_{v-1}, -D_v) \\ &= (-1, -r^2(r-\lambda)^{v-1})((r-\lambda)^{v-1}v, -r^2(r-\lambda)^{v-1}) \\ &= (-1, -1)(-1, r^2)(-1, (r-\lambda)^{v-1})((r-\lambda)^{v-1}, r^2) \\ &\quad ((r-\lambda)^{v-1}, -(r-\lambda)^{v-1})(v, r^2)(v, -(r-\lambda)^{v-1}) \\ &= (-1, (r-\lambda)^{v-1})(v, -(r-\lambda)^{v-1}) \\ &= (-1, (r-\lambda)^{v-1})(v, -1)(v, (r-\lambda))^{v-1}.\end{aligned}$$

Also

$$\prod_{j=1}^{v-2}(D_j, -D_{j+1}) = \prod_{1}^{v-2}((r-\lambda)^j(j+1), -(r-\lambda)^{j+1}(j+2))$$

$$= \left\{\prod_{1}^{v-2}((r-\lambda)^j, -(r-\lambda)^{j+1})(j+1, -(j+2))\right\} S$$

$$= S\prod_{1}^{v-2}((r-\lambda)^j, -(r-\lambda)^j)((r-\lambda)^j, (r-\lambda))(j+1, j+2)(j+1, -1)$$

$$= S\prod_{1}^{v-2}((r-\lambda), (r-\lambda))^j(j+2, -1)(j+1, -1)$$

$$= S\prod_{1}^{v-2}(r-\lambda, -1)^j(j+2, -1)(j+1, -1)$$

$$= S(r-\lambda, -1)^{(v-1)(v-2)/2}((v-1)!, -1)(v!, -1)$$

$$= S(r-\lambda, -1)^{(v-1)(v-2)/2}(v, -1),$$

where

$$S = \prod_{1}^{v-2} ((r-\lambda)^j, j+2)((r-\lambda)^{j+1}, j+1)$$
$$= \prod_{1}^{v-2} ((r-\lambda)^j, j+2)((r-\lambda)^{j-1}, j+1)$$
$$= \prod_{1}^{v-2} ((r-\lambda)^j, j+2) \prod_{j=0}^{v-3}((r-\lambda)^j, j+2)$$
$$= (r-\lambda, v)^{v-2}.$$

$$\therefore C_p(Q) = (r-\lambda, -1)^{v(v-1)/2} (v, -1)^2 (r-\lambda, v)^{2v-3}$$

$$(22) \qquad = (r-\lambda, -1)^{v(v-1)/2}(r-\lambda, v)^{2v-3}.$$

Hence we can enunciate the following theorem:

THEOREM 2. *A necessary condition for the existence of a symmetrical balanced incomplete block design with parameters v, r and λ is that*

$$C_p(Q) = (r-\lambda, -1)_p^{v(v-1)/2}(r-\lambda, v)_p^{2v-3} = +1$$

for all odd prime p, where $(m, n)_p$ is the Hilbert norm-residue symbol.

When v is even we have seen that a necessary condition for the existence of the design is that $r - \lambda$ be a perfect square. Then it is easily seen that

$$C_p(Q) = +1$$

for all odd prime p. Therefore, even if the design is really non-existent, its impossibility cannot be proved by this method.

When, however, v is odd we can in many instances demonstrate the impossibility of the design.

Consider the design

$$(A_7) \qquad v = b = 29, \quad r = k = 8, \quad \lambda = 2.$$

$$C_p(Q) = (6, -1)_p^{29.14}(6, 29)_p^{55}$$
$$= (3, 29)_p(2, 29)_p$$
$$= (29/3) \text{ for p } = 3$$
$$= (2/3) \text{ for p } = 3$$
$$= -1 \quad \text{ for p } = 3.$$

Hence the design (A_7) is impossible. As mentioned in the introduction, the impossibility has already been demonstrated by Hussain [4] by a rather lengthy method amounting to a complete exhaustion of all possibilities. The following designs with $\lambda \leq 5$ and $r, k \leq 20$ can be similarly proved to be impossible by applying Theorem 2.

(A_8)	$v = b = 137$	$r = k = 17$	$\lambda = 2$
(A_9)	$v = b = 67$	$r = k = 12$	$\lambda = 2$
(A_{10})	$v = b = 103$	$r = k = 18$	$\lambda = 3$
(A_{11})	$v = b = 53$	$r = k = 13$	$\lambda = 3$
(A_{12})	$v = b = 43$	$r = k = 15$	$\lambda = 5$
(A_{13})	$v = b = 77$	$r = k = 20$	$\lambda = 5.$

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