INDEPENDENCE OF QUADRATIC FORMS IN NORMALLY CORRELATED VARIABLES[1]

By Yukiyosi Kawada

Tokyo University of Literature and Science

The problem to give a necessary and sufficient condition that two quadratic forms in normally correlated variables are independent was treated by many authors [1], [2], [3], [4], [5]. We shall give here also a solution of this problem, which may be a generalization of that given by B. Matérn [6] for nonnegative quadratic forms to the general case.

Theorem 1. *If two quadratic forms*

$$Q_1 = \sum_{ij=1}^{n} a_{ij}x_ix_j, \quad Q_2 = \sum_{ij=1}^{n} b_{ij}x_ix_j \tag{1}$$

in normally correlated variables $x_1, \cdots, x_n$ *with zero means and with the variance matrix* I *satisfy the following four conditions*

$$F_{ij} = E(Q_1^iQ_2^j) - E(Q_1^i)E(Q_2^j) = 0 \qquad (i, j = 1, 2), \tag{2}$$

then the relation

$$AB = 0 \qquad (A = (a_{ij}), B = (b_{ij})) \tag{3}$$

holds.

Corollary 1. *If* Q_1, Q_2 *in* (1) *satisfy the four conditions* (2), *then* Q_1 *and* Q_2 *are independent.*

Corollary 2. (Necessity portion of the theorem of Craig) *A necessary condition for the independence of* Q_1 *and* Q_2 *is* $AB = 0$. (The sufficiency was proved by Craig.)

Proof of Theorem 1. The proof is very simple. Using the values $E(x_k^i) = 0$, $(i = 1, 3, 5, 7)$, $E(x_k^2) = 1$, $E(x_k^4) = 3$, $E(x_k^6) = 15$, $E(x_k^8) = 105$ $(k = 1, \cdots, n)$, we have by a straightforward calculation[2] the following relations

(4) $F_{11} = 2Tr(AB)$,

(5) $F_{12} = 8Tr(AB^2) + 4Tr(AB)Tr(B)$,

(6) $F_{21} = 8Tr(A^2B) + 4Tr(AB)Tr(A)$,

(7) $F_{22} = 32Tr(A^2B^2) + 16Tr((AB)^2) + 16Tr(AB^2)Tr(A) + 16Tr(A^2B)Tr(B) + 8Tr(AB)Tr(A)Tr(B) + 8Tr(AB)^2$.

[1] Presented at the Chapel Hill meeting of the Institute of Mathematical Statistics and Biometric Society March 18, 1950.

[2] If we apply an orthogonal transformation on $(x_1, \cdots, x_n)$ so that A becomes a diagonal form, the calculation becomes simpler than with the general form. We may note here also the fact that we need not assume that $x_1, \cdots, x_n$ are normally correlated, but we use only the values of $E(x_k^i)$ $(i = 1, \cdots, 8)$ for our proof.

Put $C = AB$. Let C' be the transposed matrix of C. We have from (2), (4)–(7)

$$2Tr(A^2B^2) + Tr((AB)^2) = 2Tr(CC') + Tr(C^2) = 0. \tag{8}$$

The left side of (8) is equal to $\sum_{i,j=1}^{n} (c_{ij}^2 + c_{ij}c_{ji} + c_{ji}^2)$, which is positive unless all $c_{ij} = 0$ $(i, j = 1, \cdots, n)$. Hence we have $C = AB = 0$, q.e.d.

Corollary 1 follows from Theorem 1 and the theorem of Craig. Corollary 2 results from observing that independence of Q_1 and Q_2 implies (2).

B. Matérn proved, that if A, B are nonnegative, then $AB = 0$ follows from a unique condition $F_{11} = 2Tr(AB) = 0$. If only one of the matrices A, B is assumed to be nonnegative, we have

THEOREM 2. *Let A be nonnegative. Then from two conditions $F_{11} = 0$, $F_{12} = 0$ in (2) follows the relation $AB = 0$.*

PROOF. From (4), (5) follows $Tr(AB^2) = 0$. Since A is nonnegative, we can choose a real symmetric matrix A_0 such that $A = A_0^2$. Put $C_0 = A_0B$. Then we have $Tr(AB^2) = Tr(C_0C_0') = 0$ and from this follows $C_0 = 0$. Hence we have also $AB = A_0C_0 = 0$, q.e.d.

REFERENCES

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[6] B. MATÉRN, "Independence of non-negative quadratic forms in normally correlated variables", *Annals of Math. Stat.*, Vol. 20 (1949), pp. 119–120.

ERRATA TO "CONTROL CHART FOR LARGEST AND SMALLEST VALUES"

BY JOHN M. HOWELL

Los Angeles City College

In the paper cited in the title (*Annals of Math. Stat.*, Vol. 20 (1949), p. 306), there are some numerical errors in Table I. Values of $d_2/2$ and d_4 are given by H. J. Godwin in "Some Low Moments of Order Statistics" in the same issue