# REMARK ON SEPARABLE SPACES OF PROBABILITY MEASURES

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Early writers on mathematical statistics often had to assume that the distributions under consideration either admitted probability densities, sometimes subject to further regularity conditions, or that they were purely discrete; in general, two separate arguments were needed to deal with the two cases. More recent authors however have achieved greater generality and, at the same time, a unification of methods by dispensing altogether with assumptions on the distributions themselves and specifying, instead, their relation to each other. In particular, these writers assume (for example in [1], [2], [3]) that the probability measures under consideration form what is sometimes called a "dominated set of measures", defined as follows: Let $X$ be the sample space, $\mathfrak{B}$ a Borel field of some subsets of $X$ and let $\Omega = \{m\}$ be a set of probability measures defined on $\mathfrak{B}$. $\Omega$ is called a dominated set of measures if there exists a $\sigma$-finite measure $\mu$ such that every $m$ in $\Omega$ is absolutely continuous with respect to $\mu$.

One of the important consequences of assuming that $\Omega$ be dominated is that, if such an $\Omega$ is metrized by introducing

$$d(m, m') = \sup_{B \epsilon \mathfrak{B}} | m(B) - m'(B) |$$

as a metric and $\mathfrak{B}$ is a separable Borel field (as for instance in the case of Borel sets in finite dimensional Euclidean spaces), then $\Omega$ is separable with respect to the topology induced by $d$. (Proof of this can be based on Hopf's approximation theorem as indicated in [1]; a proof for measures dominated by Lebesgue measure is referred to at the end of [4].)

Since the separability of dominated sets of measures is used to great advantage (for example in [1] and in [4]), one wonders whether there exist any other separable sets of measures than dominated ones. It will be shown to the contrary, that an even weaker separability condition than the one described implies that the set be dominated. In order to state the exact theorem, we shall consider a set $M = \{m\}$ of probability-measures defined on a common Borel field $\mathfrak{B}$ of subsets of some abstract space $X$ and introduce a weak topology into $M$ in the usual way (see [5]) by defining a complete system of neighborhoods as follows: For every $p$ in $M$ and for every finite collection of sets $B_1, B_2, \cdots, B_k$ in $\mathfrak{B}$ and every $\epsilon > 0$, let $\alpha = (B_1, B_2, \cdots, B_k; \epsilon)$ and let

$$V_\alpha(p) = \{m \text{ in } M \,|\, | m(B_i) - p(B_i) | < \epsilon, i = 1, 2, \cdots, k,\},$$

i.e. the set of all those measures in $M$ whose values assumed on the sets $B_1$, $B_2, \cdots, B_k$ differ less than $\epsilon$ in absolute value from the corresponding values of $p$. $V_\alpha(p)$ is called the neighborhood of index $\alpha$ of the measure $p$. By letting $\alpha$ range over all possible finite collection of sets in $\mathfrak{B}$ and all positive numbers $\epsilon$, $V_\alpha(p)$ defines a complete system of neighborhoods (see for instance [6]), so that $M$ may be regarded as a topological space. We shall prove the following theorem:

THEOREM. *If a set of measures $M$ is separable with respect to the weak topology defined above then $M$ is dominated.*

PROOF. By assumption, there exists a sequence of measures $\{m_i\}$ in $M$ such that to any given $p$ in $M$ and any given $\alpha$, there exists an $m_i$ in $V_\alpha(p)$. Let $\mu = \sum_{i=1}^{\infty} c_i m_i$, $0 < c_i < 1$, $\sum_{i=1}^{\infty} c_i = 1$; then $\mu(X) = 1$. Let $B$ in $\mathfrak{B}$ be such that $\mu(B) = 0$. Obviously, $m_i(B) = 0$ for all $i$. Let $p$ be an arbitrary fixed measure in $M$ and consider the sequence of neighborhoods $V_{\alpha_j}(p)$, where $\alpha_j = \left(B; \frac{1}{2^j}\right)$ $j = 1, 2, \cdots$. Then for any fixed $j$ there exists an $m_k$ which is in $V_{\alpha_j}(p)$, thus

$$| m_k(B) - p(B) | < \frac{1}{2^j}.$$

Since $m_i(B) = 0$ for all $i$, $p(B) < \frac{1}{2^j}$ and since $j$ was arbitrary this means $p(B) = 0$. Thus whenever $\mu(B) = 0$ for some $B$ we have $p(B) = 0$ for every $p$ in $M$, as we wanted to prove.

Since a set of measures separable with respect to the metric topology induced by

$$d(m, m') = \sup_{B \in \mathfrak{B}} | m(B) - m'(B) |$$

is a fortiori separable in the weak topology, we can add the following theorem:

THEOREM. *A necessary and sufficient condition for a set of measures defined over a separable Borel field to be separable with respect to the topology induced by the metric $d$ is that it be a dominated set of measures.*

## REFERENCES

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