TESTING PROPORTIONALITY OF COVARIANCE MATRICES[1]

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1. Summary. The problem of comparing the proportionality of covariance matrices often arises in genetic experiments. Knowledge of nonproportionality of covariance matrices is useful in selection work and in genetic interpretations. In developing a test of significance for this contrast, the likelihood ratio criterion was used. Likelihood ratio tests were obtained for two sets and for three sets of independent variance-covariance matrices. The test for r independent covariances was indicated and some unsolved problems were cited.

2. Introduction. Tests of significance of variances from normally distributed variates are available for testing the equality of:

(i) Two independent variances (Snedecor's F, Fisher's z, Mahalanobis' x [3], and Fisher and Yates' variance ratio),

(ii) k independent variances (Chi-square tests by Stevens [6], Bartlett [1], and Cochran [2]),

(iii) Two variances with unknown correlation (Pitman [5] and Morgan's test [4] and Wilks' likelihood-ratio test [8]),

(iv) k variances and of the associated covariances (Wilks' likelihood-ratio test [8]),

(v) The variances and covariances within each of several sets and the covariances between sets (Likelihood-ratio tests by Votaw [7]),

but no tests of significance are available for comparing the proportionality of two or more variance-covariance matrices.

The hypothesis of proportionality of variance-covariance matrices is more tenable than equality in many genetic experiments, since it is known that the variances are unequal but it is not known if the variance or covariance for one strain is merely a multiple of that for the other strain. Knowledge of this is of importance in any genetic study on the inheritance of characters and in selection work. In addition, the means and variances are often related in some manner and a transformation of the data may not be advisable since this may lead to incorrect genetic interpretations.

3. Likelihood ratio for comparing two covariance matrices. The problem of testing the hypothesis that the variances and covariances from strain A are proportional to the variances and covariances of strain B was solved by an

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application of the likelihood-ratio test. Let the characters be represented by $X_1, X_2, \cdots, X_p$ for strain A and by $Y_1, Y_2, \cdots, Y_p$ for strain B, respectively. The hypothesis, then, is that the variance or covariance of A equals K times the corresponding variance or covariance for B, that is, $(\sigma_{ij})_x = K(\sigma_{ij})_y$, where K is a proportionality factor and the sample variance-covariance matrices for A and B are independently estimated.

The likelihood ratio for the above in general terminology is

$$\lambda = f(a_{ij}, b_{ij}, \bar{K}, \bar{\sigma}_y^{ij})/f(a_{ij}, \hat{\sigma}_x^{ij})\, f(b_{ij}, \hat{\sigma}_y^{ij}),$$

where

$$a_{ij} = \sum_{u=1}^{n} (X_{iu} - \bar{x}_i)(X_{ju} - \bar{x}_j),\ b_{ij} = \sum_{v=1}^{m} (Y_{iv} - \bar{y}_i)(Y_{jv} - \bar{y}_j),$$

X_{iu} and Y_{iv} are the sample elements and $\bar{x}_i$ and $\bar{y}_i$ are the sample means, $i, j = 1, 2, \cdots, p$, $\bar{K}$ and $\bar{\sigma}_y^{ij}$ are the maximum likelihood estimates of K and σ_y^{ij} computed under the hypothesis that $(\sigma_{ij})_x = K(\sigma_{ij})_y$, $\hat{\sigma}_x^{ij}$ and $\hat{\sigma}_y^{ij}$ are the maximum likelihood estimates of σ_x^{ij} and σ_y^{ij} computed under the hypothesis of independence, and where there are $n - 1$ degrees of freedom associated with the a_{ij} and $m - 1$ with the b_{ij}.

It is known that the sums of squares and cross products of p normally distributed variates follow the Wishart distribution with $n - 1$ degrees of freedom. Furthermore, the joint distribution of two independent sums of squares and cross products may be written as

$$f(a_{ij}, b_{ij}, \sigma_x^{ij}, \sigma_y^{ij}) = f(a_{ij}, \sigma_x^{ij})\, f(b_{ij}, \sigma_y^{ij}),$$

which is proportional to

$$| \sigma_x^{ij} |^{\frac{1}{2}(n-1)} | \sigma_y^{ij} |^{\frac{1}{2}(m-1)} | a_{ij} |^{\frac{1}{2}(n-p-2)} | b_{ij} |^{\frac{1}{2}(m-p-2)} \exp\left[-\tfrac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} (\sigma_x^{ij} a_{ij} + \sigma_y^{ij} b_{ij}) \right].$$

The maximum likelihood estimates for σ_x^{ij} and σ_y^{ij} are:

$$\| \hat{\sigma}_x^{ij} \| = \| a_{ij}/(n - 1) \|^{-1} \text{ and } \| \hat{\sigma}_y^{ij} \| = \| b_{ij}/(m - 1) \|^{-1};$$

also $(\hat{\sigma}_{ij})_x = a_{ij}/(n - 1)$ and $(\hat{\sigma}_{ij})_y = b_{ij}/(m - 1)$.

Now under the hypothesis that the variances and covariances are proportional, i.e., $(\sigma_{ij})_x = K(\sigma_{ij})_y$, the joint distribution is proportional to

$$| \sigma_y^{ij}/K |^{\frac{1}{2}(n-1)} | \sigma_y^{ij} |^{\frac{1}{2}(m-1)} | a_{ij} |^{\frac{1}{2}(n-p-2)} | b_{ij} |^{\frac{1}{2}(m-p-2)} \exp\left[-\tfrac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \sigma_y^{ij}(a_{ij}/K + b_{ij}) \right].$$

The maximum likelihood estimates of K and σ_y^{ij} are obtained from the equations

$$\bar{K} = \sum_{i=1}^{p} \sum_{j=1}^{p} \bar{\sigma}_y^{ij}\, a_{ij}/p(n - 1)$$

and

$$|| \bar{\sigma}_y^{ij} || = || (a_{ij} + \bar{K}b_{ij})/\bar{K}(n + m - 2) ||^{-1}.$$

It is possible to solve for $\bar{K}$ in the above 2 equations. For $p = 2$ the equation in $\bar{K}$ free of $\bar{\sigma}_y^{ij}$ is

$$\bar{K}^2 \begin{vmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{vmatrix} + \bar{K}\left(1 - \frac{n + m - 2}{2(n - 1)}\right)\left(\begin{vmatrix} a_{11} & a_{12} \\ b_{12} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ a_{12} & a_{22} \end{vmatrix}\right)$$
$$+ \left(1 - \frac{2(n + m - 2)}{2(n - 1)}\right)\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = 0.$$

For $p = 3$, $\bar{K}$ is obtained by solving the following equation:

$$\bar{K}^3 \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}$$
$$+ \bar{K}^2\left(1 - \frac{n + m - 2}{3(n - 1)}\right)\left(\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\right)$$
$$+ \bar{K}\left(1 - \frac{2(n + m - 2)}{3(n - 1)}\right)\left(\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\right)$$
$$+ \left(1 - \frac{3(n + m - 2)}{3(n - 1)}\right)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0.$$

For $p = 4$ and higher, the coefficients are obtained in a like manner. The number of determinants for each power of $\bar{K}$ will be the same as the coefficients in the binomial.

The proof that there is only one positive root in the polynomial in $\bar{K}$ would be obtained by proving that the sum of the determinants, associated with any power of $\bar{K}$, is positive.[2] Since the other coefficients in the polynomial are positive up to a certain point and negative thereafter, there is only one change in sign, and thus only one positive real root. The positive root is the only one of interest here since the variances are inherently positive.

The likelihood ratio for comparing the proportionality of the variances and

[2] A proof of this was first called to my attention by Isadore Blumen.

covariances for strains A and B is

$$\lambda = \bar{K}^{\frac{1}{2}p(m-1)} \mid a_{ij}/(n - 1) \mid^{\frac{1}{2}(n-1)} \mid b_{ij}/(m - 1) \mid^{\frac{1}{2}(m-1)} \cdot \mid (a_{ij} + \bar{K}b_{ij})/(n + m - 2) \mid^{-\frac{1}{2}(n+m-2)},$$

where $-2 \log \lambda$ is distributed approximately as chi-square with

$$p(p + 1) - \tfrac{1}{2}p(p + 1) - 1 = \tfrac{1}{2}p(p + 1) - 1$$

degrees of freedom when m and n are large.

4. Likelihood ratio for comparing three covariance matrices. In the event that 3 independently estimated sets of variances and covariances are compared for proportionality, the likelihood ratio is

$$\lambda = f(a_{ij}, b_{ij}, c_{ij}, \bar{K}_1, \bar{K}_2, \bar{\sigma}_z^{ij})/f(a_{ij}, \hat{\sigma}_x^{ij})f(b_{ij}, \hat{\sigma}_y^{ij})f(c_{ij}, \hat{\sigma}_z^{ij}),$$

where $c_{ij} = \sum_{w=1}^{q} (Z_{iw} - \bar{z}_i)(Z_{jw} - \bar{z}_j)$, Z_{iw} are the sample elements and $\bar{z}_i$ the sample means, $\bar{K}_1$, $\bar{K}_2$, and $\bar{\sigma}_z^{ij}$ are the maximum likelihood estimates of K_1, K_2, and σ_z^{ij} computed under the hypothesis of proportionality, that is, $(\sigma_{ij})_x = K_1(\sigma_{ij})_y = K_2(\sigma_{ij})_z$, $(\hat{\sigma}_{ij})_x$, $(\hat{\sigma}_{ij})_y$, and $(\hat{\sigma}_{ij})_z$ are the maximum likelihood estimates of $(\sigma_{ij})_x$, $(\sigma_{ij})_y$, and $(\sigma_{ij})_z$ computed under the hypothesis of independence, a_{ij}, b_{ij}, and c_{ij} have $n - 1$, $m - 1$, and $q - 1$ degrees of freedom respectively, and the a_{ij} and b_{ij} are as defined previously.

Under the hypothesis of independence the maximum likelihood estimates of σ_x^{ij}, σ_y^{ij}, and σ_z^{ij} are $\hat{\sigma}_x^{ij}$ and $\hat{\sigma}_y^{ij}$, given in Section 3, and $\hat{\sigma}_z^{ij}$ which can be obtained from the equation $\| \hat{\sigma}_z^{ij} \| = \| c_{ij}/(q - 1) \|^{-1}$.

Under the hypothesis of proportionality the maximum likelihood estimates of K_1, K_2, and σ_z^{ij} are obtained from the equations:

$$\bar{K}_2 = \bar{K}_1 \Sigma\Sigma \bar{\sigma}_z^{ij} b_{ij}/p(m - 1),$$

$$\bar{K}_2 = \Sigma\Sigma \bar{\sigma}_z^{ij}(a_{ij} + \bar{K}_1 b_{ij})/p(n + m - 2)$$

and

$$\| \bar{\sigma}_z^{ij} \| = \| (a_{ij} + \bar{K}_1 b_{ij} + \bar{K}_2 c_{ij})/\bar{K}_2(m + n + q - 3) \|^{-1}.$$

The positive roots (probably only one for each proportionality constant) for $\bar{K}_1$ and $\bar{K}_2$ which maximize the likelihood ratio are the ones used. Substituting these values in the likelihood ratio the following results:

$$\lambda = \bar{K}_2^{-\frac{1}{2}p(m+n-2)} \bar{K}_1^{\frac{1}{2}p(m-1)} \mid a_{ij}/(n - 1) \mid^{\frac{1}{2}(n-1)} \mid b_{ij}/(m - 1) \mid^{\frac{1}{2}(m-1)} \mid c_{ij}/(q - 1) \mid^{\frac{1}{2}(q-1)} \mid \bar{\sigma}_z^{ij} \mid^{\frac{1}{2}(m+n+q-3)}$$

When sample sizes are large, $-2 \log \lambda$ is distributed approximately as chi-square with $p(p + 1) - 2$ degrees of freedom.

The method for comparing r independent sets of variances and covariances follows by a simple extension of the above likelihood ratio and the solution for the $r - 1$ proportionality constants.

5. Unsolved problems. The nature of the roots for the proportionality constants requires further study. Also, likelihood ratios could be developed for comparing the proportionality of r non-independent covariance matrices under various hypotheses. A study of these tests of significance could be made in much the same way as described by Votaw [7]. Such a study is necessary before a complete understanding of this test is obtained.

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