RELATIONS BETWEEN MOMENTS OF ORDER STATISTICS[1]

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Summary. Moments of order statistics multiplied by appropriate factors are called normalized moments. These normalized moments are shown to be successive differences of normalized moments of largest order statistics.

1. Introduction. After discovering the following relations, the author learned in conversation with H. O. Hartley that similar relations had been used by him and others at the University of London. They did not, however, recognize the advantage of expressing them in terms of what are here called "normalized moments." The extreme simplicity of the relations is a direct result of this device.

2. Derivation of the relations. Let $x_{1|n} \geq x_{2|n} \geq \cdots \geq x_{n|n}$ be the order statistics from a sample of size n. Let $g_{i|n}(x)$ be defined by

$$g_{i|n}(x) = F^{n-i}(x)[1 - F(x)]^{i-1}f(x), \tag{1}$$

where $f(x)$ and $F(x)$ are, respectively, the pdf and cdf of the population from which the sample is drawn. Then, the pdf of $x_{i|n}$ is

$$nC_{i-1}^{n-1}g_{i|n}(x),$$

where $C_j^m = m!/[j!(m - j)!]$. If r is any integer such that $r \leq i$, we may write

$$\begin{aligned}[1 - F(x)]^{i-1} &= [1 - F(x)]^{r-1}[1 - F(x)]^{i-r} \\ &= [1 - F(x)]^{r-1} \sum_{j=0}^{i-r} (-1)^{i-r-j} C_j^{i-r} F^{i-r-j}(x).\end{aligned}$$

Substituting this in (1), we have

$$g_{i|n}(x) = \sum_{j=0}^{i-r} (-1)^{i-r-j} C_j^{i-r} g_{r|n-j}(x), \qquad r \leq i \leq n. \tag{2}$$

The relations (2) may be written in matrix form. By letting $C_s^r \equiv 0$ for $s > r$, and considering the expansion of $[1 - (1 - y)]^i$, it can be shown that the inverse of the $(n + 1)$ by $(n + 1)$ matrix

$$P_n = (C_j^i), \qquad i, j = 0, 1, \cdots, n,$$

is

$$P_n^{-1} = ((-1)^{i+j}C_j^i), \qquad i, j = 0, 1, \cdots, n.$$

[1] Prepared in connection with research carried out at Princeton University and sponsored by the Office of Naval Research.

We shall introduce the symbol $g'_{p\cdots q|n}(x)$ to represent the vector

$$(g_{p|n}(x), \cdots, g_{q|n}(x)).$$

Similarly let $g'_{i|p\cdots q}(x)$ represent the vector

$$(g_{i|p}(x), \cdots, g_{i|q}(x)).$$

The vector $g_{p\cdots q|n}(x)$ will be the transpose of the vector $g'_{p\cdots q|n}(x)$, etc. A similar notation will be used for vectors whose components are moments.

Using these conventions, the relations (2) can be written

$$g_{r\cdots n|n}(x) = P_{n-r}^{-1}g_{r|n\cdots r}(x), \qquad r < n,$$

or, by inverting,

$$g_{r|n\cdots r}(x) = P_{n-r}g_{r\cdots n|n}(x). \tag{3}$$

Let $\mu_{i|n}$ represent the tth moment of $x_{i|n}$. The omission of t in this designation simplifies the notation and can lead to no confusion, since all the relations between moments are independent of t. Let the quantity

$$\nu_{i|n} = \int_{-\infty}^{\infty} g_{i|n}(x)\, x^t\, dx$$

be called the *normalized* t*th moment* of $x_{i|n}$. Evidently

$$\mu_{i|n} = nC_{i-1}^{n-1}\nu_{i|n}.$$

If relations (3) are multiplied by x^t and integrated, we have, in terms of the vector notation previously introduced,

$$\nu_{r|n\cdots r} = P_{n-r}\nu_{r\cdots n|n}, \qquad r < n. \tag{4}$$

From this fundamental relation, two special cases of interest can be written down. First, by letting $r = 1$, we have

$$\nu_{1|n\cdots 1} = P_{n-1}\nu_{1\cdots n|n}.$$

Second, because of the triangular nature of P_{n-r}, we may delete the last k components of the two vectors in relation (4), make a corresponding reduction in the order of the matrix, and write

$$\nu_{r|n\cdots r+k} = P_{n-r-k}\nu_{r\cdots n-k|n}.$$

In particular, if $k = n - r - 1$, we obtain

$$\nu_{r|n-1} = \nu_{r|n} + \nu_{r+1|n}.$$

That is to say, the normalized moments for a sample of size $n - 1$ can be obtained by summing adjacent pairs of normalized moments for a sample of size n. It follows that the normalized moments for all samples of size less than n can be obtained, by the simple operation of addition, from the normalized moments for a sample of size n. By reversing the process, it is clear that if $\nu_{1|n}$, $\nu_{1|n-1}$, $\cdots$, $\nu_{1|1}$ are known, the normalized moments for all samples of size no greater than n can be determined by successive differencing, although in this case there is a progressive loss of significant figures.

CORRECTION TO "THE PROBLEM OF THE GREATER MEAN"

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In the paper mentioned in the title (*Annals of Mathematical Statistics*, Vol. 21 (1950), pp. 469–487), the paragraph on page 484 beginning "We have given no criterion . . ." is erroneous, and should be omitted. The following paragraph would then read: "Let us suppose that Ω is given by (33). Then $f^0(v)$ is admissible and minimax, by the preceding paragraph. There is, however, another reason for preferring $f^0(v)$ . . . ."

We remark that in case a point on the plane $\{\omega : m_1 = m_2\}$ is an interior point of Ω and the risk function is $\bar{r}$, then (contrary to statements in the erroneous paragraph) $f^0(v)$ possesses the following property. *If $f(v)$ is a decision function such that $f(v) \not\equiv f^0(v)$ and*

$$\sup_{\omega \varepsilon \Omega} \bar{r}(f \mid \omega) \leq \sup_{\omega \varepsilon \Omega} \bar{r}(f^0 \mid \omega)(= \tfrac{1}{2}),$$

then $\bar{r}(f^0 \mid \omega) \leq \bar{r}(f \mid \omega)$ for all ω in Ω, the inequality being strict whenever $m_1 \neq m_2$. It follows that $f^0(v)$ is the unique decision function which is admissible and minimax. A proof of this remark is contained in an unpublished paper by R. R. Bahadur entitled "A Property of the t Statistic."

ERRATA

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In the author's paper "The theory of probability distributions of points on a lattice" (*Annals of Math. Stat.*, Vol. 21 (1950), pp. 198–217), read "2 × 2 × 3" for "2 × 3 × 3" on page 211, line 22, and on page 213, Table 8, heading.

ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Oak Ridge meeting of the Institute, March 15–17, 1951)

1. **Confidence Intervals for the Mean Rate at Which Radioactive Particles Impinge on a Type I Counter. (Preliminary Report.)** G. E. ALBERT, University of Tennessee and Oak Ridge National Laboratory.