To prove (a), note that for $|t_i| < \alpha_i$, $x_1 \geq 0$, we have

$$\iint_{u_i \geq x_i} dF_n(u_1, u_2) \leq \exp(-|t_1|x_1 - |t_2|x_2) \tag{4}$$

$$\iint_{u_i \geq x_i} \exp(|t_1|u_1 + |t_2|u_2)\, dF_n(u_1, u_2) \leq M_0 \exp(-|t_1|x_1 - t_2|x_2),$$

where $\varphi_n(|t_1|, |t_2|) \leq M_0$. Such a number $M_0 = M_0(t_1, t_2)$ exists since $\{\varphi_n(|t_1|, |t_2|)\}$ converges for $|t_i| < \alpha_i$. This gives an estimate for $M_n(x_1, x_2)$, which shows that (a) holds. The Helly selection principle ([2], pp. 60–62 and 83) leads to (b). The relations (c) and (d) follow immediately from Theorem 1.

From Theorems 1 and 2 we obtain

THEOREM 3. *Let $\{F_n(x_1, x_2)\}$ be a sequence of df's and let $\{\varphi_n(t_1, t_2)\}$ be the corresponding sequence of mgf's which are all assumed to exist for $|t_i| < \alpha_i$. Then the necessary and sufficient condition for the convergence of $\{\varphi_n(t_1, t_2)\}$ for $|t_i| < \alpha_i$ is that the relations* (a) *and* (b) *of Theorem* 2 *be satisfied.*

REFERENCES

[1] W. KOZAKIEWICZ, "On the convergence of sequences of moment generating functions," *Annals of Math. Stat.*, Vol. 18 (1947), pp. 61-69.
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# A NOTE ON THE MAXIMUM VALUE OF KURTOSIS

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In "A note on skewness and kurtosis," J. E. Wilkins (*Annals of Math. Stat.* Vol. 15 (1944), pp. 333–335) gave a short and elegant proof of the inequality for skewness and kurtosis

$$\beta_2 \geq \beta_1^2 + 1. \tag{1}$$

Then he gave an upper bound, depending on the population size $N$, for the skewness:

$$\max \beta_1 = (N - 2)/(N - 1)^{\frac{1}{2}}. \tag{2}$$

Now we shall derive an upper bound for the kurtosis. It will appear that the sign "=" in (1) is valid for the upper bounds, and the two maximum values indeed arise in the same "extreme" population.

To find the maximum value of the kurtosis $\beta_2$ we consider the function $\Sigma x_i^4$ in the $x$-space, where $\Sigma x_i^2 = N$ and $\Sigma x_i = 0$. We have to maximize $\Sigma x_i^4 - \lambda \Sigma x_i^2 - \mu \Sigma x_i$. The maximizing values are given by the $N$ equations, found by differentiation with respect to $x_i$

$$4x_i^3 - 2\lambda x_i - \mu = 0, \tag{3}$$

[1] "Aspirant" of the Belgian National Foundation for Scientific Research.

together with the two relations $\Sigma x_i^2 = N$ and $\Sigma x_i = 0$. Multiplication with $x_i$ and summation over all $N$ equations give

$$4N\beta_2 - 2N\lambda = 0;$$

hence

$$\max \beta_2 = \tfrac{1}{2}\lambda.$$

Since it is not possible that all values of $x$ in the population are equal, the equation (3) of third degree must have at least two different real roots, and hence it has three real roots, which we may represent by

$$\sqrt{\frac{2\lambda}{3}} \cos \alpha, \quad \sqrt{\frac{2\lambda}{3}} \cos (\alpha + \tfrac{2}{3}\pi), \quad \sqrt{\frac{2\lambda}{3}} \cos (\alpha + \tfrac{4}{3}\pi),$$

where $\cos 3\alpha = 4 \cos^3\alpha - 3 \cos \alpha = \mu/(2\lambda/3)^{3/2}$. Suppose the number of those roots is, respectively, $k$, $l$, and $m$, with sum $N$. Writing $v$ for $l - m$, we have, since not all values in the population can be the same,

$$\begin{Bmatrix} 1 \le k \le N - 1 \\ |v| \le N - k \end{Bmatrix} \text{ or } \begin{Bmatrix} k = 0 \\ |v| \le N - 2 \end{Bmatrix},$$

with $v$ even if $N - k$ is even and $v$ odd if $N - k$ is odd. $\Sigma x_i = 0$ gives, since $\lambda \neq 0$,

$$(k - \tfrac{1}{2}l - \tfrac{1}{2}m) \cos \alpha + (\tfrac{1}{2}\sqrt{3}l - \tfrac{1}{2}\sqrt{3}m) \sin \alpha = 0,$$

$$(\tfrac{3}{2}k - \tfrac{1}{2}N) \cos \alpha + \tfrac{1}{2}\sqrt{3}v \sin \alpha = 0,$$

$$\tan \alpha = -\frac{3k - N}{v\sqrt{3}},$$

$$\sin \alpha = \frac{3k - N}{\sqrt{(3k - N)^2 + 3v^2}},$$

$$\cos \alpha = \frac{-v\sqrt{3}}{\sqrt{(3k - N)^2 + 3v^2}}.$$

Hence our second relation $\Sigma x_i^2 = N$ gives

$$\begin{aligned} \frac{3N}{2\lambda} &= k \cos^2\alpha + l(-\tfrac{1}{2} \cos \alpha + \tfrac{1}{2}\sqrt{3} \sin \alpha)^2 + m(-\tfrac{1}{2} \cos \alpha - \tfrac{1}{2}\sqrt{3} \sin \alpha)^2 \\ &= k \cos^2 \alpha + (l + m)(\tfrac{1}{4} \cos^2 \alpha + \tfrac{3}{4} \sin^2 \alpha) - (l - m)\cdot\tfrac{1}{2}\sqrt{3} \sin \alpha \cos \alpha \\ &= k \frac{3v^2}{(3k - N)^2 + 3v^2} + (N - k) \frac{\tfrac{3}{4}v^2 + \tfrac{1}{4}(3k - N)^2}{(3k - N)^2 + 3v^2} \\ &\qquad + v\cdot\tfrac{1}{2}\sqrt{3}\, \frac{(3k - N)\, v\sqrt{3}}{(3k - N)^2 + 3v^2} \qquad (4) \\ &= \frac{9}{4} k - \frac{1}{4} N - \frac{(3k - N)^3}{(3k - N)^2 + 3v^2}. \end{aligned}$$

To find the maximum kurtosis we have to find the minimum possible value of the expression (4). Partial differentiation of (4) $\frac{\partial}{\partial(3k - N)}$ gives

$$\frac{3}{4} - \frac{3(3k-N)^2}{(3k-N)^2+3v^2} + \frac{2(3k-N)^4}{\{(3k-N)^2+3v^2\}^2}$$

$$= \frac{-\frac{1}{4}(3k-N)^4 - \frac{9}{2}v^2(3k-N)^2 + \frac{27}{4}v^4}{\{(3k-N)^2+3v^2\}^2} = \frac{-\frac{1}{4}\{(3k-N)^2+9v^2\}^2 + \frac{27}{2}v^4}{\{(3k-N)^2+3v^2\}^2}$$

$$\leq \frac{-\frac{81}{4}v^4 + \frac{27}{2}v^4}{\{(3k-N)^2+3v^2\}^2} \leq 0.$$

Hence $3N/2\lambda$ for every $v$, is decreasing with increasing $k$, so that we have to take the greatest possible value of $k$. In that case $3k - N$ is certainly positive and we have to take the smallest possible value of $|v|$ to minimize $3N/2\lambda$. In virtue of the conditions of $k$ and $v$ we have the "extreme" combinations $k = N - 1, |v| = 1$ and $k = N - 2, v = 0$. Substituting in (4) gives for $\lambda$, respectively, $2(N^2 - 3N + 3)/(N - 1)$ and $N$. Since those values are equal only for $N = 2$ or 3, and

$$2\frac{N^2 - 3N + 3}{N - 1} > N$$

if $N \geq 4$, we find for our upper bound

$$\max \beta_2 = \tfrac{1}{2}\lambda = \frac{N^2 - 3N + 3}{N - 1}.$$ [2]

And indeed

$$\max \beta_2 = \max \beta_1^2 + 1.$$

[2] After writing this paper, the author derived the more general formulae:

$$\max \beta_{2n} = \frac{(N-1)^{2n+1} + 1}{N(N-1)^n}$$

and

$$\max \beta_{2n+1} = \frac{(N-1)\{(N-1)^{2n+2} - 1\}}{N(N-1)^{n+1}}.$$

Their proof will be published shortly in the Dutch mathematics and physics periodical *Simon Stevin.*