ON RATIOS OF CERTAIN ALGEBRAIC FORMS

By Robert V. Hogg

State University of Iowa

1. Introduction. In an investigation of the ratio of the mean square successive difference to the mean square difference in random samples from a normal universe with mean zero, J. D. Williams [4] proved the rather surprising fact that any moment of this ratio is equal to the corresponding moment of the numerator divided by that of the denominator. Later Tjallings Koopmans [2] and John von Neumann [3] showed independently that this ratio and its denominator are stochastically independent. From this, Williams' theorem is an immediate consequence. In this paper, we determine a necessary and sufficient condition for the stochastic independence of a ratio and its denominator. We then use this condition in our study of certain ratios of algebraic forms.

2. Stochastic independence of a ratio and its denominator. We prove the following theorem for the continuous type distribution. Consider two one-dimensional random variables x and y and their probability density function $g(x,y)$. Let $P(y \leqq 0) = 0$. Assume the moment generating function, $M(u,t) = E[\exp(ux + ty)]$, exists for $-T < u,t < T$, $T > 0$. The theorem is as follows.

Theorem 1. *Under the conditions stated, in order that y and $r = x/y$ be stochastically independent, it is necessary and sufficient that*

$$\frac{\partial^k M(0,t)}{\partial u^k} \equiv \frac{\dfrac{\partial^k M(0,0)}{\partial u^k}}{\dfrac{\partial^k M(0,0)}{\partial t^k}} \frac{\partial^k M(0,t)}{\partial t^k},$$

for $k = 0, 1, 2, \cdots$.

Proof of necessity. If $f(r, y)$ is the probability density function of the variables r and y, it is well known that a necessary and sufficient condition for the independence of the random variables r and y is that $f(r,y) \equiv f_1(r)f_2(y)$, where $f_1(r)$ and $f_2(y)$ are the marginal density functions of r and y respectively. Hence, since $x = ry$,

$$M(u,t) \equiv E[\exp(ury + ty)];$$

or

$$M(u,t) \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(ury + ty) f_1(r) f_2(y)\, dr\, dy.$$

By hypothesis, the moments of x of order k exist; so

$$\frac{\partial^k M(0,t)}{\partial u^k} \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (ry)^k \exp(ty) f_1(r) f_2(y)\, dr\, dy.$$

Finally,

$$\frac{\partial^k M(0, t)}{\partial u^k} \equiv \int_{-\infty}^{\infty} r^k f_1(r)dr \cdot \int_{-\infty}^{\infty} y^k \exp (ty) f_2(y) dy,$$

for $k = 0, 1, 2, \cdots$. If we set $t = 0$, we see that $\int_{-\infty}^{\infty} r^k f_1(r) dr$ exists, since it is equal to the quotient of the kth moments of x and y,

$$K_k = \frac{\dfrac{\partial^k M(0, 0)}{\partial u^k}}{\dfrac{\partial^k M(0, 0)}{\partial t^k}}.$$

The hypothesis precludes the moments of y being zero. We also note that

$$\int_{-\infty}^{\infty} y^k \exp (ty) f_2(y) dy \equiv \frac{\partial^k M(0, t)}{\partial t^k};$$

consequently

$$\frac{\partial^k M(0, t)}{\partial u^k} \equiv K_k \frac{\partial^k M(0, t)}{\partial t^k},$$

for $k = 0, 1, 2, \cdots$.

Proof of sufficiency. Consider the identity

$$\frac{\partial^k M(0, t)}{\partial u^k} \equiv K_k \frac{\partial^k M(0, t)}{\partial t^k},$$

or

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^k \exp (ty) g(x,y) dx\, dy \equiv K_k \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^k \exp (ty) g(x,y) dx\, dy. \tag{2.1}$$

Since all the moments of x and y exist, we may differentiate p times with respect to t under the integral signs. Then if we set $t = 0$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^k y^p g(x,y) dx\, dy = K_k \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^{k+p} g(x,y) dx\, dy,$$

for $p = 0, 1, 2, \cdots$. Although t has been restricted to the range $-T < t < T$, we may extend that range to $-\infty < t < T$ and still have the existence of $M(u,t)$. The condition that $P(y \leqq 0) = 0$ further permits us to integrate (2.1) $p' \leqq k$ times under the integral signs as shown below.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{t_{p'}} \cdots \int_{-\infty}^{t_2} x^k \exp (t_1 y) g(x, y) \prod_{j=1}^{p'} dt_j\, dx\, dy$$

$$= K_k \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{t_{p'}} \cdots \int_{-\infty}^{t_2} y^k \exp (t_1 y) g(x,y) \prod_{j=1}^{p'} dt_j\, dx\, dy,$$

or

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{x}{y}\right)^{p'} \cdot x^{k-p'} g(x,y) dx\, dy = K_k \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^{k-p'} g(x,y) dx\, dy,$$

for $p' = 1, 2, 3, \cdots, k$. These two expressions may be written

$$E(x^k y^m) = K_k E(y^{k+m})$$

for $k = 0, 1, 2, \cdots$ and $m = -k, \cdots, -1, 0, 1, 2, \cdots$. If $m = -k$, then

$$E\left[\left(\frac{x}{y}\right)^k\right] = K_k.$$

Thus

$$E(x^k y^m) = E\left[\left(\frac{x}{y}\right)^k\right] E(y^{k+m}),$$

or

$$E\left[\left(\frac{x}{y}\right)^k y^{k+m}\right] = E\left[\left(\frac{x}{y}\right)^k\right] \cdot E(y^{k+m}),$$

for $k = 0, 1, 2, \cdots$ and $m = -k, \cdots, -1, 0, 1, 2, \cdots$. This could also be rewritten as

$$E(r^k y^h) = E(r^k) \cdot E(y^h),$$

for $k = 0, 1, 2, \cdots$ and $h = 0, 1, 2, \cdots$. This is sufficient to insure stochastic independence of r and y; thus the proof is complete.

3. Ratios of linear forms in gamma variables. Let the independent random variables x_j have the gamma density functions

$$f_j(x_j) = \begin{cases} \dfrac{1}{\Gamma(c_j + 1) d_j^{c_j+1}} (x_j)^{c_j} \exp\left(-\dfrac{x_j}{d_j}\right), & 0 \leqq x < \infty, \\ 0, & \text{elsewhere,} \end{cases}$$

where $c_j > -1$ and $d_j > 0$, for $j = 1, 2, \cdots, n$. Construct the two real linear forms $L_1 = \sum_1^n a_j x_j$ and $L_2 = \sum_1^n b_j x_j$, $b_j > 0$. Let L_1 and L_2 be linearly independent; thus their ratio will not be a mere constant.

THEOREM 2. *Under the conditions stated, a necessary and sufficient condition that L_2 and L_1/L_2 be stochastically independent is that*

$$b_1 d_1 = b_2 d_2 = \cdots = b_n d_n .$$

PROOF. Our proof consists in showing, by the use of Theorem 1, that if some of the bd values are distinct, the variance of L_1/L_2 is equal to zero. This fact further implies that the ratio is a constant, and hence the necessity of the condition is proved by contradiction. For the sufficiency, we demonstrate that the partial derivatives of the moment generating function $E[\exp(uL_1 + tL_2)]$ satisfy the condition of Theorem 1. However in interest of conservation of paper, a referee[1] has suggested that upon setting $u_j^2 = x_j/d_j$, von Neumann's argument [3] may be made to complete the proof.

[1] We take this opportunity to thank the Referee for this and other suggestions.

An interesting consequence of Theorem 2 is the following corollary. Let $Q_1 = X'AX$ and $Q_2 = X'BX$ be two real symmetric quadratic forms in n random values of a variable normally distributed with mean zero. We restrict Q_2 to be nonnegative (or nonpositive). Let $AB = BA$. It is known ([1], p. 25) that there then exists an orthogonal matrix C such that simultaneously $C'AC$ and $C'BC$ are diagonal matrices formed by the characteristic numbers a_j of A and b_j of B respectively. Let the rank of AB equal the rank of A. Thus if $b_j = 0$, the corresponding $a_j = 0$. Further let Q_1 and Q_2 be linearly independent.

COROLLARY. *If the above conditions are satisfied, a necessary and sufficient condition that Q_2 and Q_1/Q_2 be stochastically independent is that $B^2 = bB$, where b is a real nonzero constant.*

This corollary is essentially the theorem suggested by von Neumann's original argument.

4. Ratios of linear forms.

THEOREM 3. *Let x have a continuous distribution such that $m(t) = E[\exp(tx)]$ exists for $-T < t < T$, $T > 0$. Let the real linear forms $L_1 = \sum_1^n a_j x_j$ and $L_2 = \sum_1^n x_j$, in n random values of x, be linearly independent. Provided $P(x \leqq 0) = 0$ $[P(x \geqq 0) = 0]$, a necessary and sufficient condition for L_2 and L_1/L_2 to be stochastically independent is that x $[-x]$ have a gamma distribution.*

PROOF OF SUFFICIENCY. We use Theorem 2. If x has a gamma distribution and the set $x_1, x_2, \cdots, x_n$ is a random sample, then $d_1 = d_2 = \cdots = d_n$. We also note that $b_1 = b_2 = \cdots = b_n = 1$. Hence $b_1 d_1 = b_2 d_2 = \cdots = b_n d_n$. This implies that L_2 and L_1/L_2 are stochastically independent.

PROOF OF NECESSITY. Write

$$\begin{aligned} M(u, t) &= E[\exp(uL_1 + tL_2)], \\ &= \prod_1^n m(a_j u + t). \end{aligned}$$

Since the conditions of Theorem 1 are satisfied, the stochastic independence of L_2 and L_1/L_2 implies

$$\frac{\partial^k M(0, t)}{\partial u^k} \equiv K_k \frac{\partial^k M(0, t)}{\partial t^k}, \qquad k = 0, 1, 2, \cdots. \tag{4.1}$$

Using this condition for $k = 1$ we find

$$\sum_1^n a_j = nK_1. \tag{4.2}$$

For $k = 2$, (4.1) becomes

$$\begin{aligned} \left(\sum_1^n a_j^2\right) [m''(t)][m(t)]^{n-1} + \left(2 \sum_{i<j} a_i a_j\right) [m'(t)]^2[m(t)]^{n-2} \\ \equiv K_2 \{n[m''(t)][m(t)]^{n-1} + n(n-1)[m'(t)]^2[m(t)]^{n-2}\}. \end{aligned} \tag{4.3}$$

We now show that this identity implies that

$$[m''(t)][m(t)]^{n-1} = c[m'(t)]^2[m(t)]^{n-2}, \tag{4.4}$$

where

$$c = \frac{m''(0)[m(0)]^{n-1}}{[m'(0)]^2[m(0)]^{n-2}}.$$

To do this we assume (4.4) is not true. That is, we assume $m''(t)[m(t)]^{n-1}$ and $[m'(t)]^2[m(t)]^{n-2}$ to be linearly independent. By considering the coefficients of the linearly independent functions in (4.3), we find

$$\sum_1^n a_j^2 = nK_2$$

and

$$2\sum_{i<j} a_i a_j = n(n-1)K_2.$$

Adding these two equations we have

$$\left(\sum_1^n a_j\right)^2 = n^2K_2.$$

This result with (4.2) implies that $K_1^2 = K_2$. However $K_1 = E[L_1/L_2]$ and $K_2 = E[(L_1/L_2)^2]$; so the variance of the ratio must equal zero. This requires the ratio to equal a constant; that is, $K_1 = L_1/L_2$. However this is contrary to the hypothesis that L_1 and L_2 be linearly independent. Thus (4.4) must be an identity.

We have now found that the stochastic independence of L_2 and L_1/L_2 imposes the restriction

$$m''(t)\, m(t) = c[m'(t)]^2$$

on the moment generating function of the distribution from which the samples are drawn. Since $m(t)$ is a moment generating function, $m(0) = 1$, $m'(0) = E(x)$, and $m''(0) = E(x^2)$. Moreover, with a continuous distribution, $E(x^2) > [E(x)]^2$ and hence $c > 1$. Accordingly, we can say that (4.1) for $k = 1, 2$ requires $m(t)$ to be the unique solution to the above differential equation with the given boundary condition $m(0) = 1$. That is,

$$m(t) = (1 - bt)^{1/(1-c)}, \qquad c > 1,$$

where b is an arbitrary constant. Hence (4.1) for $k = 1, 2$ restricts us to moment generating functions of the gamma type. It might be urged that (4.1) for $k = 3, 4, 5 \cdots$ could further restrict our solution. But this can not be the case since we proved the sufficiency of the gamma distribution for the stochastic independence of L_2 and L_1/L_2. That is, $M(u,t)$ must satisfy (4.1) if $m(t) = E[\exp(tx)]$, where x has a gamma distribution. This completes the proof of the necessity of the condition.

The author wishes to express his appreciation to Professor A. T. Craig for the suggestions made during the preparation of this paper.

REFERENCES

[1] H. Weyl, *The Theory of Groups and Quantum Mechanics*, Methuen and Co., Ltd., London, 1931.

[2] Tjalling Koopmans, "Serial correlation and quadratic forms in normal variables," *Annals of Math. Stat.*, Vol. 13 (1942), pp. 14–33.

[3] J. von Neumann, "Distribution of the ratio of the mean square successive difference to the variance," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 367–395.

[4] J. D. Williams, "Moments of the ratio of the mean square successive difference to the mean square difference in samples from a normal universe," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 239–241.

[5] Robert V. Hogg, "On ratios of certain algebraic forms in statistics," unpublished thesis, State University of Iowa.