quotient of ranges in samples from a rectangular population," *Jour. Am. Stat. Assn.*, Vol. 46 (1951), pp. 502–507), who also gives the correct density of the ratio for $R \geqq 1$. The correct cumulative distribution for $R \geqq 1$ is

$$1 - R^{-n_2}\left\{\frac{Rn_2n_1(n_1 - 1)}{(n_1 + n_2 - 1)(n_1 + n_2 - 2)} - \frac{n_1(n_1 - 1)(n_2 - 1)}{(n_1 + n_2)(n_1 + n_2 - 1)}\right\}.$$

ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Blacksburg meeting of the Institute, March 19–21, 1952)

1. On the Approximation of Sampling Distributions by Punch Card Methods.

Carl F. Kossack and Lester L. Helms, Purdue University.

This paper presents a procedure for obtaining empirical distributions, by punch card methods, of statistics for which the exact distribution or a usable approximation has not been found. The mechanization of random sampling of a univariate population has been described and extended to random sampling of a correlated multivariate population whose covariance matrix is given. This procedure has been applied to Wald's classification statistic in the univariate case, and the results noted.

2. Resolvable Incomplete Block Designs with Two Replications.

R. C. Bose and K. R. Nair, University of North Carolina.

Incomplete block designs in which the blocks can be grouped in such a way that each group contains a complete replication may be called resolvable designs. They are useful from the point of view of recovery of inter-block information. It is therefore important to investigate resolvable designs involving a few replications. In this paper we consider a class of resolvable designs with two replications, which contains as a special case the well known square and rectangular lattices with two replications. Given a symmetrical balanced incomplete block design with u treatments, and r replications in which each pair occurs λ times, we can use the incidence matrix (n_{ij}) of this design to form a design of one class in the following way. Take a $u \times u$ square scheme, and in the cell (i, j) put x new treatments when $n_{ij} = 1$, and y new treatments when $n_{ij} = 0$. The total number of treatments obtained in this way is $v = u[rx + (u - r)y]$. The design is now constructed by taking the rows of the scheme for the blocks of the first replication, and the columns of the scheme for the blocks of the second replication. It has been shown that both the intra- and inter-block analysis can be carried out in a simple manner. The necessary formulae have been given, and the computational procedure illustrated by working out a numerical example.

3. Rank Analysis of Incomplete Block Designs. I. The Method of Paired Comparisons.

R. A. Bradley and M. E. Terry, Virginia Polytechnic Institute.

True preferences or ratings $\pi_{1u}, \cdots, \pi_{tu}, \Sigma_{i=1}^{t} \pi_{iu} = 1$, are assumed to exist for t treatments in the uth of g groups of experimental data in an experiment involving paired comparisons. For the uth group, the probability that treatment i is "better" than treatment j when they appear in a pair is postulated to be $\pi_{iu}/(\pi_{iu} + \pi_{ju})$.

Three tests of hypotheses are available and estimates of the treatment ratings may be

obtained. The tests use likelihood ratio statistics to test (a) $H_0 : \pi_{iu} \equiv 1/t$, against $H_1 : \pi_{iu} \equiv \pi_i$ for all u; (b) $H_0 : \pi_{iu} \equiv 1/t$, against $H_1 : \pi_{iu} \not\equiv 1/t$; and (c) $H_0 : \pi_{iu} \equiv \pi_i$ for all u, against $H_1 : \pi_{iu} \not\equiv \pi_i$.

Small-sample distributions with tables are available for tests (a) and (b). In all three tests limiting distributions are shown to be in the form of chi-square.

4. Multiple Regression with a Quantal Response. D. B. DUNCAN AND R. C. RHODES, Virginia Polytechnic Institute.

The problem considered is that of fitting a maximum likelihood multiple regression equation to data in which the response is quantal, the probit transformation is appropriate and the number, r, of independent regression variates is not small.

Iterative methods, for example the Bliss-Fisher method, are available, but these have been developed mainly for the case $r = 1$ and rapidly become impractical for cases $r > 2$.

A method is developed based on (i) the approximation of the weighted deviations of the working probits from the provisional probits by linear functions of the provisional probits and (ii) the replacement of the independent x variates throughout most of the procedure by a linear function of them, termed a composite regression variate. These devices lead to a simple procedure and result in an estimated 70 to 90% saving in work.

5. Rank Analysis of Incomplete Block Designs. II. The Method for Blocks of Three. (Preliminary Report.) R. A. BRADLEY AND M. E. TERRY, Virginia Polytechnic Institute.

The extensions of "Rank analysis of incomplete block designs. I. The method of paired comparisons," Abstract No. 3 above, to blocks of size three are presented. As before, true preferences or ratings $\pi_{1u}, \cdots, \pi_{tu}$, $\Sigma_{i=1}^{t} \pi_{iu} = 1$ are assumed to exist for t treatments in the uth of g groups. For the uth group the probability that treatment i obtains top ranking in the presence of treatments j and k is $\pi_{iu}/(\pi_{iu} + \pi_{ju} + \pi_{ku})$ and the probability that treatment j obtains rank 2, given that i had rank 1, is $\pi_{ju}/(\pi_{ju} + \pi_{ku})$.

The three test of hypotheses listed in the first paper are again developed. Tables are under preparation but are not yet available or complete.

6. Limit Theorems Associated with Variants of the von Mises Statistic. M. ROSENBLATT, University of Chicago.

A multidimensional analogue of the von Mises statistic is considered for the case of sampling from a multidimensional uniform distribution. The limiting distribution of the statistic is shown to be that of a weighted sum of independent chi-square random variables with one degree of freedom. The weights are the eigenvalues of a positive definite symmetric function. A modified statistic of the von Mises type useful in setting up a two-sample test is shown to have the same limiting distribution under the null hypothesis (both samples come from the same population with a continuous distribution function) as that of the one-dimensional von Mises statistic. The paper makes use of elements of the theory of stochastic processes.

7. A Modification of Schwarz's Inequality with Applications to Distributions. SIGEITI MORIGUTI, University of North Carolina and University of Tokyo.

Let $\Phi(t)$ be a function of bounded variation in the closed interval $[a, b]$ and continuous at both ends. Then for any nondecreasing function $x(t)$ belonging to $L_2(a, b)$, and summable with respect to Φ, $\int_a^b x(t)\, d\Phi(t) \leqq \left\{\int_a^b x(t)^2\, dt\right\}^{\frac{1}{2}} \left\{\int_a^b \bar{\phi}(t)^2\, dt\right\}^{\frac{1}{2}}$, where $\bar{\phi}(t)$ is the right-hand derivative of the "greatest convex minorant" of $\Phi(t)$. This is proved and necessary and sufficient conditions for the equality to hold are also given. Several examples of application to distribution problems in statistics are discussed.

8. Confidence Intervals of Fixed Geometric Size for Scale Parameters. (Preliminary Report.) LIONEL WEISS, University of Virginia.

A procedure is given for obtaining confidence intervals for parameters of scale with confidence coefficient no less than β and length no greater than Δ, where β is any number between 0 and 1 and Δ is any positive number. The procedure uses two samples, the size of the second sample being a chance variable. It seems certain that there are other procedures for the same purpose yielding a smaller expected number of observations, but even in using the method given the problem of fixing the size of the first sample to minimize the expected number of observations is tedious computationally. A comparison is suggested between the expected number of observations and the number of observations required when an upper bound for the scale parameter is known and a single sample is used to get a confidence interval of at least a given confidence coefficient and of length bounded by a given number.

9. On Lower Bounds of Powers of Certain Multivariate Tests. S. N. ROY, University of North Carolina.

For multivariate normal populations tests of hypotheses were earlier offered for (i) equality of two covariance matrices; (ii) independence of two set of variates, and (iii) the analysis of variance situation. Lower bounds of the powers of such tests are now discussed. Here, for simplicity, under (iii) is considered the hypothesis of equality of respective means for k p-variate populations with a common covariance matrix Σ_2. Let S_1 denote the "covariance matrix of the sample means," S_2 the "pooled covariance matrix of sample error," Σ_1 the corresponding population matrix of means, H_0 the hypothesis (iii), and H an alternative. Then the critical region of the test at a level α is: $\theta_q \geqq \theta_0$, where θ_0 is given by $P(\theta_q \geqq \theta_0 \mid H_0) = \alpha$ and θ_q is the largest characteristic root of the matrix $S_1 S_2^{-1}$ (positive semidefinite of rank $q \equiv \min(p, k - 1)$, a.e.). For the power we have the following lower bound:

$$P(\theta_q \geq \theta_0 \mid H) > 1 - \prod_{i=1}^{q} \{1 - P(\text{noncentral } F \geq \theta_0 \mid \Theta_i)\},$$

the noncentral F being with d.f. $(k - 1)$ and $(N - k)$ (N: total number of observations), and Θ_i's being the characteristic roots of the matrix $\Sigma_1 \Sigma_2^{-1}$ (positive semidefinite of rank, say, $s \leqq q$). Similar lower bounds are also readily available for (i) and (ii).

10. Normal Multivariate Analysis and the Orthogonal Group. A. T. JAMES, Princeton University.

The relationship of the orthogonal group, and its two coset spaces, the Grassmann and Stiefel manifolds, to normal multivariate sampling theory is discussed. The use of the Blaschke differential forms to represent the invariant measures on the two manifolds is illustrated by a derivation of the well known distribution of the canonical correlation coefficients in the null case. The distribution of n independent samples from a normal k-variate population is transformed into 3 independent distributions, viz., (a) essentially the Wishart distribution; (b) the distribution of the linear subspace spanned by the sample when represented as k vectors in n-space; this is given by the invariant measure in the Grassmann manifold; (c) the invariant distribution of a $k \times k$ orthogonal matrix which determines the orientation of the k vectors in the k-dimensional linear subspace.

11. Exact Formulae in Sequential Analysis for Exponential Distributions. JOHAN H. B. KEMPERMAN, Purdue University.

Let $a > 0$ and $b > 0$. Let $X_1, X_2, \cdots$ be a sequence of independent random variables with a common distribution (we assume $Pr\ (X_i \neq 0) > 0$). Put $Z_n = X_1 + X_2 + \cdots X_n$ and let N be the random variable which takes the value n if $-a < Z_k < b (k = 1, \cdots, n - 1)$ and $Z_n \geqq b$ or $Z_n \leqq -a$. We put $\rho_n = Pr\ (N = n)$, $p_n = Pr\ (N = n, Z_n \leqq -a)$ and $q_n = Pr\ (N = n, Z_n \geqq b)$. Let D be an open connected region in the complex z-plane containing an interval G on the imaginary axis. We suppose that there exists a function $\psi(t)$ which is analytic in D and which in G takes the value $\phi(t) = E(e^{tX})$. Then, the function which for t in G is defined by $r_n(t) = \rho_n E(e^{tZ_n} \mid N = n)$ can be extended to an analytic function $r_n(t)$ in D. Moreover, there exists a constant $\theta (0 < \theta < 1)$ such that for each value t in D with $\mid \phi(t) \mid \geqq \theta$ we have $\Sigma_1^\infty r_n(t)\ (t)^{-n} = 1$ (Wald's fundamental identity). For the same values t, this relation may be differentiated term by term with respect to t. This generalization is used to obtain generating functions for p_n and q_n under certain conditions.

12. A Note on a Generalized Behrens-Fisher Problem. HENRY SCHEFFÉ, Columbia University.

An exact solution [HENRY SCHEFFÉ, "On solutions of the Behrens-Fisher problem, based on the t-distribution," *Annals of Math. Stat.*, Vol. 14 (1943), pp. 35–44; "A note on the Behrens-Fisher problem," *Annals of Math. Stat.*, Vol. 15 (1944), pp. 430–432] of the Behrens-Fisher problem, based on the t-distribution, is generalized to yield confidence intervals for a linear combination of unknown parameters.

13. Large-Sample Confidence Intervals for Density Function Values at Percentage Points. JOHN E. WALSH, China Lake, California.

Let us consider a sample of size n from a population with density function $f(x)$. Let θ_p represent the $100p\%$ point of this population. A class of "well behaved" density functions is defined. This class seems to contain density functions which are capable of approximating most practical situations of a continuous type for $.05 \leqq p \leqq .95$. This paper presents some approximate confidence intervals for $f(\theta_p)$ for the case where $.05 \leqq p \leqq .95$ and the density function is of the "well behaved" class. These results hold for values of n which are only moderately large. The exact value of a confidence coefficient is not known but is determined within reasonably close limits. An approximate expression is obtained for deciding when n is sufficiently large for application of these results. The minimum sample sizes required depend on p and the confidence coefficient; they range from around fifty to several thousand. The confidence intervals are based on statistics of the form $x[(p + \epsilon)n + C\sqrt{n}] - x[(p - \epsilon) - C\sqrt{n}]$, where $x[z] = x$[integer nearest z] and $x[1], \cdots, x[n]$ are the sample values arranged in increasing order of magnitude. The quantity ϵ is a small but fixed number depending on p, while C is chosen so that a confidence interval of the desired order of magnitude is obtained.

14. Sequential Sufficient Statistics. R. R. BAHADUR, Delhi, India.

The author defines sequential sufficiency and gives some characterizations of it. Let $x_1, x_2, \cdots$ be a sequence of abstract chance variables having a joint distribution p belonging to a family P of probability distributions. For each m let $X_{(m)}$ be the space of all points $(x_1, x_2, \cdots, x_m)$, and let t_m be a function on $X_{(m)}$ with arbitrary range such that t_m is a sufficient statistic for P when the sample space is $X_{(m)}$. Then $(t_1, t_2, \cdots)$ is said to be a sequential sufficient statistic if for any event A depending only on $x_1, x_2, \cdots$ and x_m the conditional probability of A given t_{m+1} equals the conditional expectation given t_{m+1} of the conditional probability of A given t_m, $(m = 1, 2, \cdots)$. The role of sequential sufficient statistics in sequential decision problems has been described elsewhere [RAGHU RAJ BAHADUR, "On sufficiency and statistical decision functions," *Annals of Math. Stat.*, Vol. 22 (1951), pp. 609–610 (abstract)]. The main result established here is the following. If $x_1, x_2, \cdots$ and x_m are independently distributed and their joint distribution is absolutely continuous with respect to a fixed σ-finite measure λ_m, $(p \varepsilon P;\ m = 1, 2, \cdots)$, then $(t_1, t_2, \cdots)$ is a sequential sufficient statistic.

15. **Some Powerful Rank Order Tests.** WASSILY HOEFFDING, University of North Carolina.

It is shown that in certain cases there exist nonparametric tests which depend only on the ranks of the observations and whose power is arbitrarily close to the power of a standard parametric test if the sample is sufficiently large. For example, let $(x_1, y_1), \cdots, (x_n, y_n)$ be a random sample from a continuous bivariate distribution. Let H be the hypothesis that x and y are independent. Let r_i and s_i be the respective ranks of x_i and y_i. Let $h_n(k)$ be the expected value of the kth order statistic in a sample of n observations from a normal (0, 1) distribution. Let $c_n = \Sigma_{i=1}^{n} h_n(r_i) h_n(s_i)$. Let k_n be the smallest number for which the probability of $| c_n | > k_n$ does not exceed α when H is true. Suppose that (x, y) has a bivariate normal distribution with correlation ρ (which may depend on n), and that the power of the standard product-moment correlation test of size α tends to a constant $\beta \leqq 1$ as $n \to \infty$. Then the power of the test which rejects H if $| c_n | > k_n$ tends to the same limit β. Similar results hold for two-sample tests, analysis of variance tests, etc. (Work sponsored by the Office of Naval Research.)

16. **Confidence Bounds for a Set of Means.** D. A. S. FRASER, University of Toronto.

The following problem was suggested to the author by Professor John Tukey: given $x_1, \cdots, x_n$ are normal and independent with means $\mu_1, \cdots, \mu_n$ and variance σ^2, to find an upper confidence bound (or confidence interval) for the set of means $\mu_1, \cdots, \mu_n$. This paper proves that, subject to mild restrictions on the type of bound, exact β-level confidence bounds (or intervals) do not exist (unless $n = 1$ or $\beta = 0, 1$). Incidental to the proof, bounds are obtained having at least β confidence: they are $\max x_i + \lambda_{1-\beta}\sigma$ for the upper bound and $(\min x_i + \lambda_{1-\frac{1}{2}(1-\beta)}\sigma, \max x_i + \lambda_{\frac{1}{2}(1-\beta)}\sigma)$ for the interval, where λ_α is the value exceeded with probability α by a standardized normal variate.

If the μ's are values of the location parameter for a distribution with density $f_\mu(x) = f(x - \mu)$, then a bound (interval) with at least β confidence is obtained by using the above formulas with $\sigma = 1$ and with α defined as the α point of the distribution having $\mu = 0$. If this class of distributions is bounded complete with respect to the location parameter μ (using at least all μ less than, say, zero), then exact upper bounds to not exist.