$N = N_1N_2$. On the other hand, the second component is taken direct'yfrom the array $B = (b_{ij})$.

Now select any t rows from the array so constructed. Any t-plet of the b elements is repeated N_2 times in each of λ_2 groups. Within each of these groups of N_1 objects any particular t-plet of the a elements occurs λ_1 times so that each t-plet which is constructed from the compound elements occurs $\lambda_1\lambda_2$ times. Thus the new array is orthogonal.

We now adjoin the array (N_3, k_3, s_3, t), where $k = \min(k_1, k_2, k_3)$, to the one we have just constructed, by an analogous process. Continuing in this manner, we reach our theorem. In particular if $t = 2$, and $\lambda_i = 1$ for $i = 1, 2, \cdots, u$, we secure the MacNeish theorem (cf. [1]).

As an example of the use of our theorem, we can state as an illustrative result

$$f(72, 6, 2) \geqq 4$$

since $f(3^2, 3, 2) = 4$, $f(2^3, 2, 2) = 7$ in accordance with results established in [4]. In the absence of this extension of the MacNeish result, it might have been supposed that there could be but three orthogonal rows for this case, since there are no orthogonal Latin squares of side 6. We cannot, however, conclude that the equality sign holds since counter examples have been given in [4].

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ON A LIMITING CASE FOR THE DISTRIBUTION OF EXCEEDANCES, WITH AN APPLICATION TO LIFE-TESTING

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According to equation (4.12) of [1], the probability that in a future sample of N observations, taken from an unknown distribution of a continuous variate, less than x of them will exceed x_m, the mth highest observation in the trial sample of n observations, is given by

$$W(n, m, N, x) = 1 - \frac{\binom{N}{x+1}}{\binom{N+n}{x+1}} F_m(x + 1, -n, -n - N + x + 1, 1),$$

where F_m is the sum of the first m terms of the hypergeometric series, having the parameters indicated in the parentheses. If we set $m = 1$, we find that the probability of getting in a future sample of N trials at most x exceedances of the largest value in a trial sample of n observations is

$$W(n, 1, N, x) = 1 - \left[\binom{N}{x+1} \Big/ \binom{N+n}{x+1}\right], \tag{1}$$

since $F_1 = 1$.

If x and N are both large, we can approximate the factorials in (1) with Stirling's formula, $a! \approx \sqrt{2\pi a}(a/e)^a$. Then (1) reduces to

$$\begin{aligned} 1 - W(n, 1, N, x) &\approx \left(1 - \frac{x+1}{N}\right)^n \\ &\cdot \left\{\frac{\left(1 + \frac{n}{N-x-1}\right)^{N-x-1+n}}{\left(1+\frac{n}{N}\right)^{n+N}}\right\} \sqrt{1 - \frac{n}{n+N}} \sqrt{1 + \frac{n}{N-x-1}}. \end{aligned} \tag{2}$$

Now consider the limiting case in which N and x both approach infinity in such a way that $x = kN$. This is the case in which we wish to find the probability that in a very large future sample at most a fraction k of the observations will exceed the largest value in the trial sample of n observations. Considering each of the factors on the right side of (2), we have

$$\lim_{x=kN\to\infty} \left(1 - \frac{x+1}{N}\right)^n = (1-k)^n,$$

$$\lim_{x=kN\to\infty} \left(1 + \frac{n}{N-x-1}\right)^{N-x-1+n} = \lim_{x=kN\to\infty} \left(1 + \frac{n}{N}\right)^{N+n} = e^n,$$

$$\lim_{x=kN\to\infty} \sqrt{1 - \frac{n}{N+n}} = \lim_{x=kN\to\infty} \sqrt{1 + \frac{n}{N-x-1}} = 1.$$

Hence,

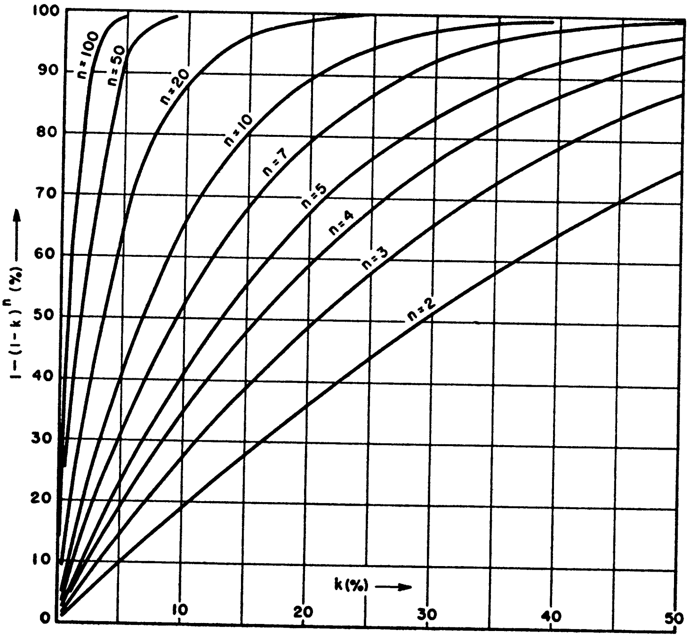
$$\lim_{N\to\infty} W(n, 1, N, kN) = 1 - (1-k)^n. \tag{3}$$

The probability density, which may be obtained from (3) by differentiation, is

$$p(k) = n(1 - k)^{n-1}. \tag{4}$$

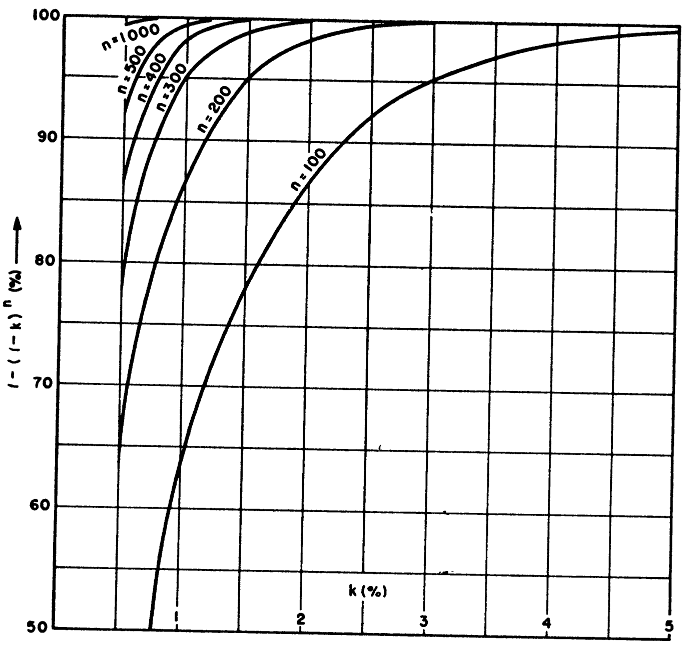
An interesting check on the consistency of the theory is a proof of (4) based on Gumbel's original discrete distribution of x. Setting $m = 1$ in equation (1.3) of [1] we have

$$w(n, 1, N, x) = \frac{n}{N+n} \left\{\frac{\binom{N}{x}}{\binom{N+(n-1)}{x}}\right\}.$$



100
90
80
70
60
50
40
30
20
10
0
$1-(1-k)^n$ (%)
n=100
n=50
n=20
n=10
n=7
n=5
n=4
n=3
n=2
k(%)
10
20
30
40
50

FIG. 1



100
90
80
70
60
50
$1-(1-k)^n$ (%)
n=1000
n=500
n=400
n=300
n=200
n=100
k(%)
1
2
3
4
5

FIG. 2

Note that the factor in brackets is the same as the last term on the right side of (1) with n replaced by $n - 1$ and $x + 1$ replaced by x. Hence for large x and N,

$$w(n, 1, N, x) \approx \frac{n}{N + n}\left(1 - \frac{x}{N}\right)^{n-1}$$

$$\cdot \left\{\frac{\left(1 + \dfrac{n}{N - x}\right)^{N-x+n}}{\left(1 + \dfrac{n - 1}{N}\right)^{N+n-1}}\right\} \sqrt{1 - \frac{n - 1}{N + n - 1}} \sqrt{1 + \frac{n - 1}{N - x}}.$$

By the same limiting procedure as before,

$$\lim_{x=kN\to\infty} w(n, 1, N, kN) = \frac{n}{N}(1 - k)^{n-1}. \tag{5}$$

In any small interval dk, there are Ndk possible values that x can assume; hence the probability that k lies in the interval dk is

$$p(k)\, dk = \frac{n}{N}(1 - k)^{n-1}(N\, dk). \tag{6}$$

Therefore, $p(k) = n(1 - k)^{n-1}$. This is exactly the result given by equation (4), but obtained in a somewhat different way.

From the symmetry of the problem, $\lim_{N\to\infty} W(n, 1, N, kN)$ is also the probability that in a large future sample at most a fraction k of the observations will be *less* than the *smallest* observation in the original trial sample of n units. Hence, a life-test of n units may be discontinued as soon as any unit fails and equation (3) will give the probability that in the future at most $100k\%$ of the units will fail in a time shorter than the length of the test. The graphs show W as a function of k for various values of n.

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CORRECTION TO "THE SAMPLING DISTRIBUTION OF THE RATIO OF TWO RANGES FROM INDEPENDENT SAMPLES"

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In the note mentioned in the title (*Annals of Math. Stat.*, Vol. 21 (1950), pp. 112–116) the distribution given for the above mentioned ratio when the sample values are drawn from a rectangular distribution is correct only when $R \leqq 1$. This is pointed out in an article by P. R. Rider ("The distribution of the