| DESIGNATING NUMBER | $\frac{m_1}{\sigma}$ | $\frac{m_2}{\sigma}$ |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 0 | 1 |
| 3 | 1 | 0 |
| 4 | 1 | 1 |
| 5 | 0 | 2 |
| 6 | 2 | 0 |
| 7 | 2 | 2 |

Selected ordinates for systematic and randomized procedures for these 7 pairs of values are presented and compared in Table 1. It is seen that the tails of some of the curves are much heavier than for case 1 ($m_1 = m_2 = 0$), indicating that much larger values of $F$ are required for significance. On the other hand, some of the tails are lighter than for case 1 so that smaller $F$-values are indicative of significance at the usual levels. Randomization is effective in some cases in giving a distribution that is closer to the conventional $F$ distribution than is the $F$ distribution for a systematic procedure.

It is easy to find the limiting values of the ratios of the ordinates of (2.12) and (2.13) to the ordinates of the conventional $F$ distribution as $F$ approaches 0 and $\infty$ (same). These limiting values are also indicated in Table 1.

When (2.13) is a greatly curtailed distribution making errors of the first kind less probable than expected then the probability of errors of the second kind may be greatly enhanced.

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# A GENERALIZATION OF A THEOREM DUE TO MacNEISH[1]

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**1. Summary and introduction.** In 1922 MacNeish [1] considered the problem of orthogonal Latin squares and showed that if the number $s$ is written in standard form:

$$s = p_0^{n_0} p_1^{n_1} \cdots p_k^{n_k},$$

[1] This note is a revision of one section of the author's doctoral dissertation submitted to the University of North Carolina at Chapel Hill.

where $p_0, p_1, \cdots, p_k$ are primes, and if

$$r = \min(p_0^{n_0}, p_1^{n_1}, \cdots, p_k^{n_k}),$$

then we can construct $r - 1$ orthogonal Latin squares of side $s$. An alternative proof was also given by Mann [2]. At the April, 1950 meeting of the Institute of Mathematical Statistics at Chapel Hill, North Carolina, R. C. Bose announced an interesting generalization of this result [3] which is stated as a theorem in the next section. The proof given here is simpler than Bose's original proof and is published at his suggestion.

**2. Bose's generalization of MacNeish's theorem.** Let us consider a matrix $A = (a_{ij})$, where each $a_{ij}$ represents one of the integers $0, 1, \cdots, s - 1$ with $N$ columns and $k$ rows. Consider all $t$-rowed submatrices of $N$ columns which can be formed from this array by choosing any $t$ rows. Each column of the submatrix can be regarded as an ordered $t$-plet. The matrix $A$ will be called an orthogonal array $(N, k, s, t)$ of size $N$, $k$ constraints, $s$ levels, strength $t$ and index $\lambda$ if each of the $C_t^k$ $t$-rowed submatrices that may be formed from $A$ contains every one of the $s^t$ possible ordered $t$-plets each repeated $\lambda$ times. It is clear that we cannot add rows indefinitely to the array and still preserve its orthogonal character. We shall use the symbol $f(N, s, t)$ to denote the maximum number of constraints that are possible.

THEOREM. *If $N_i$ is divisible by $s_i^t$ for $i = 1, 2, \cdots, u$, then*

$$f(N_1N_2 \cdots N_u, s_1s_2 \cdots s_u, t) \geqq \min(k_1, k_2, \cdots, k_u),$$

*where $k_i = f(N_i, s_i, t)$.*

PROOF. Let $N_i = \lambda_i s_i^t$. We shall proceed inductively, and we first establish the relationship:

$$f(N_1N_2, s_1s_2, t) \geqq \min(k_1, k_2).$$

Let us denote the orthogonal array with $N_1$ columns and $k_1$ constraints by $A = (a_{ij})$ and the second array with $N_2$ columns and $k_2$ constraints by $B = (b_{ij})$. We may regard the elements of these two arrays as elements of two additive Abelian groups. Accordingly we may form the direct sum of these two groups. There are $s_1s_2$ elements in this sum, and we may represent any element of this new group by the symbol $(a_{ij}, b_{mn})$ where $a_{ij}$ and $b_{mn}$ are elements of the two modules. We now write the array with $N_1N_2$ columns in the form

$$\begin{array}{ccccccc}
(a_{k1}, b_{k1}) & \cdots & (a_{kN_1}, b_{k1}) & \cdots & (a_{k1}, b_{kN_2}) & \cdots & (a_{kN_1}, b_{kN_2}) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(a_{11}, b_{11}) & \cdots & (a_{1N_1}, b_{11}) & \cdots & (a_{11}, b_{1N_2}) & \cdots & (a_{1N_1}, b_{1N_2}),
\end{array}$$

where the elements of $A$ are used for the first component for the first $N_1$ columns and for the first $k$ rows, where $k = \min(k_1, k_2)$. The construction is completed in a similar manner for the next group of $N_1$ columns (not indicated in the array above) and so on until $N_2$ groups of $N_1$ columns have been written down so that

$N = N_1N_2$. On the other hand, the second component is taken direct'yfrom the array $B = (b_{ij})$.

Now select any $t$ rows from the array so constructed. Any $t$-plet of the $b$ elements is repeated $N_2$ times in each of $\lambda_2$ groups. Within each of these groups of $N_1$ objects any particular $t$-plet of the $a$ elements occurs $\lambda_1$ times so that each $t$-plet which is constructed from the compound elements occurs $\lambda_1\lambda_2$ times. Thus the new array is orthogonal.

We now adjoin the array $(N_3, k_3, s_3, t)$, where $k = \min(k_1, k_2, k_3)$, to the one we have just constructed, by an analogous process. Continuing in this manner, we reach our theorem. In particular if $t = 2$, and $\lambda_i = 1$ for $i = 1, 2, \cdots, u$, we secure the MacNeish theorem (cf. [1]).

As an example of the use of our theorem, we can state as an illustrative result

$$f(72, 6, 2) \geqq 4$$

since $f(3^2, 3, 2) = 4$, $f(2^3, 2, 2) = 7$ in accordance with results established in [4]. In the absence of this extension of the MacNeish result, it might have been supposed that there could be but three orthogonal rows for this case, since there are no orthogonal Latin squares of side 6. We cannot, however, conclude that the equality sign holds since counter examples have been given in [4].

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## ON A LIMITING CASE FOR THE DISTRIBUTION OF EXCEEDANCES, WITH AN APPLICATION TO LIFE-TESTING

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According to equation (4.12) of [1], the probability that in a future sample of $N$ observations, taken from an unknown distribution of a continuous variate, less than $x$ of them will exceed $x_m$, the $m$th highest observation in the trial sample of $n$ observations, is given by

$$W(n, m, N, x) = 1 - \frac{\binom{N}{x+1}}{\binom{N+n}{x+1}} F_m(x + 1, -n, -n - N + x + 1, 1),$$