SOME RANK ORDER TESTS WHICH ARE MOST POWERFUL AGAINST SPECIFIC PARAMETRIC ALTERNATIVES[1]

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Summary. The most powerful rank order tests against specific parametric alternatives are derived.

Following the methods of Hoeffding [4], we derive the most powerful rank order test of whether N observations come from the same but unknown population against the alternative that the observations $Z_1, \cdots, Z_N$ come from populations which have the joint density

$$\prod_{i=1}^{N} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(z_i - d_i\xi - \eta)^2\right],$$

where $d_1, \cdots, d_N$ are given constants, not all equal, and ξ/σ is sufficiently small. The test criterion was found to be $c_1(R) = \sum d_i EZ_{N,r_i}$, where EZ_{Ni} is the expected value of the ith standard normal order statistic and $R = (r_1, \cdots, r_N)$ is the permutation of the ranks. The distribution of this statistic was shown to be asymptotically normal providing the known constants $d_1, \cdots, d_N$ satisfied Noether's condition [9].

The two-sample distribution is a special case, and the resultant statistic $c_1(R)$ is shown to be asymptotically normal. The approximation of the distribution of the $c_1(R)$ statistic to the distribution $C(1 - x^2)^{\frac{1}{2}N-2}$, $-1 \leqq x \leqq 1$, is investigated. This statistic is then compared to the existing Mann and Whitney U statistic. No method having been found for analytical evaluation of the power of this test, the power was examined experimentally.

Tables are appended giving the exact distribution of the $c_1(R)$ statistic for all possible subsample sizes whose total size is less than or equal to 10 together with the corresponding Mann and Whitney U value. Table 2 gives critical values of $c_1(R)$ for $N \leqq 10$, $p \leqq .10$.

1. Introduction. The problem of testing whether two samples come from the same population when either there are observed measurements and no assumption about the functional form of the underlying distributions or there are only ranks of the observations available has been treated by many statisticians.

H. Scheffé [12] gives an exhaustive and succinct review of this problem together with brief descriptions of proposed solutions offered previous to 1943. Included in Scheffé's references are the well known "runs" test and the modified likelihood ratio test of Wald and Wolfowitz [14], [17].

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[1] This paper was presented at the Boston meeting of the Institute on December 26, 1951. These results were included in the thesis submitted to the Dept. of Mathematical Statistics of the University of North Carolina in partial fulfillment of the requirements for the degree of Doctor of Philosophy, June 1951.

A rank analogue to the standard parametric test was introduced in 1945 by Wilcoxon [16], and a linear function of Wilcoxon's T was proposed by K. K. Matthen [7] in 1946 and its distribution and property of consistency shown by Mann and Whitney [6] in 1947. Tables are given by Mann and Whitney [6] for values of $m, n \leqq 8$.

Implicit in the introduction to Tables XX and XXI of Fisher and Yates [3] is a consideration by Fisher and Yates of rank order tests of the type discussed and developed in this dissertation. W. Hoeffding [4] develops a general most powerful rank order test, and this paper is an application of his results to specific problems, with various extensions.

2. Derivation of the statistic $c_1(R)$. Let $Z = (Z_1, \cdots, Z_N)$ be a random vector of N components with probability function $P(S) = P\{Z \varepsilon S\}$, and W be the set of points $(z_1, \cdots, z_N)$ in N-dimensional Euclidean space where $z_i \neq z_j$, $i \neq j$. We shall consider only probability functions such that

$$P(W) = 1. \tag{2.1}$$

Define D_0 to be a class of probability functions $P(S)$ which are invariant under all the $N!$ permutations of the Z_i and H_0 to be the hypothesis that $P(S)$ is in D_0. In particular H_0 may be the hypothesis that $Z_1, Z_2, \cdots, Z_N$ are independent with a common continuous distribution.

Let $R = (r_1, \cdots, r_N)$ be a permutation of the integers $(1, \cdots, N)$. Let $S(R)$ be the subset of W where z_i has rank r_i, $(i = 1, \cdots, N)$. Let the integer t_i be defined by $r_{t_i} = i$, $(i = 1, \cdots, N)$. Then $S(R)$ is the set where $z_{t_1} < z_{t_2} < \cdots < z_{t_N}$. The set W consists of the $M = N!$ subsets $S(R)$. For the probability $P(S(R))$ of the set $S(R)$ we shall use the short notation $P(R)$. A probability function $P(S)$ which is in D_0 will be denoted by $P(S| H_0)$. Let H_1 be the hypothesis that $P(A) = P_1(A)$, a probability function of Z not in D_0. Denote the M permutations R by $R_1, \cdots, R_M$ in such a way that

$$P_1(R_i) \geqq P_1(R_j) \qquad \text{if } i = 1, \cdots, m; j = m + 1, \cdots, M.$$

Clearly, $(R_1, \cdots, R_m)$ determines a rank order test which is most powerful for testing H_0 against H_1 [4].

If the condition (2.1) is not satisfied, we may apply either of the following two rules.

Rule 1. If n of the values $Z_1, \cdots, Z_N$ are less than Z_j, and m values are equal to Z_j, the latter values are assigned at random the ranks $n + 1, \cdots, n + m$. This is carried out for all Z_j, and Z is treated as belonging to the corresponding set $S(R)$.

Rule 2. Let k be the number of sets $S(R)$ which can be obtained by applying 1 to a given Z. If exactly k' of these belong to the critical region, H_0 is rejected with probability k'/k.

It follows from the definition of D_0 that

$$P(R \mid H_0) = \frac{1}{N!} \tag{2.2}$$

for all R.

Consider the alternative where $Z_1, \cdots, Z_N$ have the joint density

$$\prod_{i=1}^{N} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(z_i - d_i\xi - \eta)^2\right], \tag{2.3}$$

where $d_1, \cdots, d_N$ are given constants, not all equal, and ξ, η are parameters. Then

$$P(R) = \underset{z_{t_1} < \cdots < z_{t_N}}{\int \cdots \int} \prod_{i=1}^{N} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(z_i - d_i\xi - \eta)^2\right] dz_1 \cdots dz_N. \tag{2.4}$$

The probability $P(R)$ is independent of η and depends on $\delta = \xi/\sigma$ only. For if we make the transformation

$$z_i' = \frac{1}{\sigma}(z_i - \eta), \qquad (i = 1, \cdots, N),$$

the inequalities $z_{t_1} < z_{t_2} < \cdots < z_{t_N}$ are transformed into $z'_{t_1} < z'_{t_2} < \cdots < z'_{t_N}$. Denoting $P(R)$ by $P(R, \delta)$, and replacing z_i' by z_i, we have

$$P(R, \delta) = \underset{z_{t_1} < \cdots < z_{t_N}}{\int \cdots \int} \prod_{i=1}^{N} f(z_i - d_i\delta)\, dz_1 \cdots dz_N, \tag{2.5}$$

where

$$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right). \tag{2.6}$$

It is easily seen that $P(R, \delta)$ has continuous derivatives of any order with respect to δ. Hence we can write, for any positive integer k,

$$P(R, \delta) = \frac{1}{N!}\left[c_0(R) + c_1(R)\frac{\delta}{1} + \cdots + c_k(R)\frac{\delta^k}{k!} + O(\delta^{k+1})\right], \tag{2.7}$$

where

$$\frac{1}{N!} c_\eta(R) = \left.\frac{d^\eta(P(R, \delta))}{d\delta^\eta}\right|_{\delta=0},$$

and $O(\delta^{k+1})$ denotes a term of order δ^{k+1} as $\delta \to 0$. We shall also write formally

$$P(R, \delta) = \frac{1}{N!}\sum_{\eta=0}^{\infty} c_\eta(R)\frac{\delta^\eta}{\eta!} \tag{2.8}$$

without considering the convergence of the series.

To obtain the coefficients $c_\eta(R)$ we shall expand the integrand in (2.5) in powers of δ and integrate term by term.

$$f(z_i - d_i\delta) = \sum_{\eta_i=0}^{\infty} \frac{f^{(\eta_i)}(z_i)(-d_i\delta)^{\eta_i}}{\eta_i!}, \tag{2.9}$$

where $f^{(\eta_i)}(z_i)$ is the η_ith derivative of $f(z_i)$. Then

$$\prod_{i=1}^{N} f(z_i - d_i\delta) = \prod_{i=1}^{N} \sum_{\eta_i=0}^{\infty} \frac{f^{(\eta_i)}(z_i)(-d_i\delta)^{\eta_i}}{\eta_i!}$$

$$(2.10) \qquad = \sum_{\eta_1=0}^{\infty} \cdots \sum_{\eta_N=0}^{\infty} \frac{(-\delta) \exp\left[\sum_{i=1}^{N} \eta_i\right]}{\eta_1!\,\eta_2! \cdots \eta_N!} \prod_{i=1}^{N} f^{(\eta_i)}(z_i)\, d_i^{\eta_i}$$

$$= \sum_{\eta=0}^{\infty} \sum\nolimits^{(\eta)} \frac{\eta!}{\prod_{i=1}^{N} \eta_i!} \prod_{i=1}^{N} d_i^{\eta_i} f^{(\eta_i)}(z_i) \frac{(-\delta)^{\eta}}{\eta!},$$

where $\sum^{(\eta)}$ represents the summation over all $\eta_1, \cdots, \eta_N$ such that $\eta_i \geqq 0$, $i = 1, \cdots, N$, and $\sum_{i=1}^{N} \eta_i = \eta$. Integrating the final form of (2.10) term by term and comparing the resultant expansion with (2.8) we obtain

$$(2.11) \quad c_\eta(R) = N! \int_{z_{t_1} < \cdots < z_{t_N}} \cdots \int \sum\nolimits^{(\eta)} \frac{\eta!}{\prod_{i=1}^{N} \eta_i!} \prod_{i=1}^{N} d_i^{\eta_i}(-1)^{\eta_i} f^{(\eta_i)}(z_i)\, dz_1 \cdots dz_N.$$

Consider the following transformation. Let $z_{t_i} = z_i'$, $i = 1, \cdots, N$, and now the domain of integration will be $(z_1' < \cdots < z_N')$. Since by definition $t_{r_i} = i$, $i = 1, \cdots, N$, where r_i is the rank of z_i in $S(R)$, we have $z_i = z_{t_{r_i}} = z_{r_i}'$. Applying this transformation to $c_\eta(R)$ in (2.11) and replacing z_i' by z_i, we have

$$(2.12) \quad c_\eta(R) = N! \int_{z_1 < \cdots < z_N} \cdots \int \sum\nolimits^{(\eta)} \frac{\eta!}{\prod_{i=1}^{N} \eta_i!} \prod_{i=1}^{N} (-d_i)^{\eta_i} f^{(\eta_i)}(z_{r_i})\, dz_1 \cdots dz_N.$$

But

$$(2.13) \qquad f^{(\eta)}(z) = (-1)^{\eta} H_\eta(z) f(z),$$

where $H_\eta(z)$ is the ηth order Hermite polynomial [1]. In particular, $H_0(z) = 1$, $H_1(z) = z$, $H_2(z) = z^2 - 1$. Substituting the appropriate forms of (2.13) in (2.12) we have

$$(2.14) \quad c_\eta(R) = N! \int_{z_1 < \cdots < z_N} \cdots \int \sum\nolimits^{(\eta)} \frac{\eta!}{\prod_{i=1}^{N} \eta_i!} \prod_{i=1}^{N} d_i^{\eta_i} H_{\eta_i}(z_{r_i}) f(z_{r_i})\, dz_1 \cdots dz_N.$$

Let $Z_{N1} \leqq Z_{N2} \leqq \cdots \leqq Z_{NN}$ be the order statistics in a random sample of N from a standard normal population. The joint frequency function of $Z_{N1}, \cdots, Z_{NN}$ is $N! \prod_{i=1}^{N} f(z_i)$, $z_1 < \cdots < z_N$. Hence

$$(2.15) \qquad c_\eta(R) = E\left[\sum\nolimits^{(\eta)} \frac{\eta!}{\prod_{i=1}^{N} \eta_i!} \prod_{i=1}^{N} d_i^{\eta_i} H_{\eta_i}(Z_{Nr_i})\right].$$

In particular,

$$c_0(R) = 1, \tag{2.16}$$

$$c_1(R) = \sum_{i=1}^{N} d_i E(Z_{Nr_i}), \tag{2.17}$$

$$\begin{aligned} c_2(R) &= E\left[\sum_{i=1}^{N} d_i Z_{Nr_i}\right]^2 - \sum_{i=1}^{N} d_i^2, \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} d_i d_j E(Z_{Nr_i} Z_{Nr_j}) - \sum_{i=1}^{N} d_i^2. \end{aligned} \tag{2.18}$$

To the order of δ^3, $P(R, \delta)$ may be written as follows:

$$P(R, \delta) = \frac{1}{N!}\left[1 + c_1(R)\delta + c_2(R)\frac{\delta^2}{2} + O(\delta^3)\right]. \tag{2.19}$$

If δ is positive and sufficiently small, the rank order test of size $K/N!$ which is most powerful against $P(S, \delta)$ is determined by the K permutations R_i, $i = 1, \cdots, K$, for which $c_1(R)$ takes its largest values. Hence $c_1(R)$ is the desired statistic for the stated alternative with δ positive and small.

3. Asymptotic distribution. It will be proved that under H_0 the statistic $c_1(R)$ is asymptotically normally distributed as $N \to \infty$ if the d_i satisfy a certain condition. The proof is based on a theorem of Noether [9] and an unpublished theorem of Hoeffding which, together with its proof, is reproduced here by permission.

A sequence of sequences $A_N = (a_{N1}, \cdots, a_{NN})$, $N = 1, 2 \cdots$ is said to satisfy Condition W if as $N \to \infty$,

$$\frac{\frac{1}{N}\sum_{i=1}^{N}(a_{Ni} - \bar{a}_N)^r}{\left[\frac{1}{N}\sum_{i=1}^{N}(a_{Ni} - \bar{a}_N)^2\right]^{r/2}} = O(1), \qquad r = 3, 4, \cdots, \tag{3.1}$$

where $\bar{a}_N = 1/N \sum_{i=1}^{N} a_{Ni}$ [15]. Similarly a sequence of sequences satisfies Noether's condition [9] if

$$\frac{\sum(a_{Ni} - \bar{a}_N)^r}{[\sum(a_{Ni} - \bar{a}_N)^2]^{r/2}} = o(1), \qquad r = 3, 4, \cdots. \tag{3.2}$$

Let $D_N = (d_{N1}, \cdots, d_{NN})$ and $H_N = (h_{N1}, \cdots, h_{NN})$, $N = 1, 2, \cdots$, be two sequences of sequences of real numbers. Let $(x_1, \cdots, x_N)$ be the random vector whose domain consists of the $N!$ equally probable permutations of $(h_{N1}, \cdots, h_{NN})$, and let

$$L_N = d_{N1}x_1 + \cdots + d_{NN}x_N,$$

$$L_N^0 = \frac{L_N - E_0(L_N)}{\sqrt{V_0(L_N)}},$$

where $E_0(L_N)$ and $V_0(L_N)$ denote the mean and the variance of L_N.

Noether [9] shows that

$$E_0(L_N) = \frac{1}{N}\sum_{i=1}^{N} d_{Ni}\sum_{i=1}^{N} h_{Ni} = N\bar{d}_N\bar{h}_N, \tag{3.3}$$

$$V_0(L_N) = \frac{1}{N-1}\sum_{i=1}^{N}(d_{Ni}-\bar{d}_N)^2\sum_{i=1}^{N}(h_{Ni}-\bar{h}_N)^2, \tag{3.4}$$

and that if the sequence H_N satisfies Condition W, and the D_N satisfies Noether's condition, L_N^0 has a normal limiting distribution with mean 0 and variance 1.

The statistic $c_1(R)$ (2.17) can be written

$$c_1(R) = \sum_{i=1}^{N} d_i h_{r_i},$$

where

$$h_i = h_{Ni} = E(Z_{Ni}). \tag{3.5}$$

(The $d_i = d_{Ni}$ may also depend on N; to simplify the notation we write d_i, h_i for d_{Ni}, h_{Ni}). Under H_0 the $N!$ permutation of $(h_1, \cdots, h_N)$ are equally probable, so that $c_1(R) = L_N$, with h_{Ni} defined by (3.5).

Hoeffding shows that the sequences $H_N = (h_1, \cdots, h_N) = (EZ_{N1}, \cdots, EZ_{NN})$ satisfy Condition W as follows.

It is easily seen from the symmetry of the normal distribution that $h_i = EZ_{Ni} = -EZ_{N,N+1-i} = -h_{n+1-i}$. Hence

$$\sum_{i=1}^{N} h_i^r = 0 \text{ if } r \text{ is odd.} \tag{3.6}$$

In particular $\bar{h}_N = 0$. Hence it suffices to show that

$$\frac{\frac{1}{N}\sum_{i=1}^{N} h_i^{2k}}{\left[\frac{1}{N}\sum_{i=1}^{N} h_i^2\right]^k} = O(1) \qquad \text{for } k = 2, 3, \cdots.$$

If $r = 2k$ is even, we have

$$\sum_{i=1}^{N} h_i^{2k} = \sum_{i=1}^{N}(EZ_{Ni})^{2k} \leqq \sum_{i=1}^{N} E(Z_{Ni})^{2k} = E\sum_{i=1}^{N} Z_{Ni}^{2k}, \tag{3.7}$$

and

$$E\sum Z_{Ni}^{2k} = NEX^{2k}, \tag{3.8}$$

where X is standard normal. Equation (3.8) holds since a symmetrical function of the order statistics is distributed as the same function of the unordered independent variables.

If $m = [N/2]$ is the largest integer $\leqq N/2$, we have by Cauchy's inequality

$$(3.9) \qquad \sum_{i=1}^{N} h_i^2 = 2 \sum_{i=N-m+1}^{N} h_i^2 \geqq \frac{2}{m}\left(\sum_{i=N-m+1}^{N} h_i\right)^2 = \frac{2}{m}\left(\sum_{i=N-m+1}^{N} EZ_{Ni}\right)^2.$$

Let $F(x) = \int_{-\infty}^{x} f(y)\, dy$. Then for $1 \leqq k \leqq N - 1$,

$$(3.10) \qquad \begin{aligned} \sum_{i=k+1}^{N} EZ_{Ni} &= \sum_{i=k+1}^{N} N\binom{N-1}{i-1} \int_{-\infty}^{\infty} xf(x)[F(x)]^{i-1}[(1 - F(x))]^{N-1}\, dx \\ &= N \int_{-\infty}^{\infty} xf(x) \sum_{j=k}^{N-1} \binom{N-1}{j} [F(x)]^j[1 - F(x)]^{N-1-j}\, dx. \end{aligned}$$

Since $\sum_{j=k}^{N-1}\binom{N-1}{j}[F(x)]^j[1 - F(x)]^{N-1-j}$ is the remainder after k terms of the binomial $(F(x) + 1 - F(x))^{N-1}$, we may express [5] this remainder as the incomplete Beta function

$$\frac{(N-1)!}{(k-1)!(N-1-k)!} \int_0^{F(x)} t^{k-1}(1-t)^{N-1-k}\, dt.$$

Hence

$$(3.11) \qquad \sum_{i=k+1}^{N} EZ_{Ni} = N \int_{-\infty}^{\infty} xf(x) \frac{(N-1)!}{(k-1)!(N-1-k)!} \int_0^{F(x)} t^{k-1}(1-t)^{N-1-k}\, dt\, dx.$$

Integrating by parts we obtain

$$(3.12) \qquad \sum_{i=k+1}^{N} EZ_{Ni} = \frac{N!}{(k-1)!(N-k-1)!} \int_{-\infty}^{\infty} [f(x)]^2[(Fx)]^{k-1}[1 - F(x)]^{N-1-k}\, dx.$$

Since $F(-x) = 1-F(x) \leqq x^{-1}f(x)$ for $x > 0$, there exists a constant $K > 0$ such that

$$(3.13) \qquad f(x) \geqq KF(x)[1-F(x)] \text{ for all } x.$$

For let $c > 0$; then

$$d = \min_{x<c} \frac{f(x)}{F(x)[1 - F(x)]} > 0.$$

For $x > c$, $f(x) > x[1-F(x)] \geqq x[1-F(x)][F(x)] \geqq cF(x)[1-F(x)]$; and by symmetry the same inequality holds for $x < c$. Hence we can take $K = \min(c, d)$.

$$(3.14) \qquad \begin{aligned} \sum_{i=k+1}^{N} EZ_{Ni} &\geqq K \frac{N!}{(k-1)!(N-1-k)!} \int_{-\infty}^{\infty} f(x)[F(x)]^k[1 - F(x)]^{N-k}\, dx \\ &= K\frac{k(N-k)}{N+1}. \end{aligned}$$

Whence from (3.9)

$$\sum_{i=1}^{N} h_i^2 \geqq \frac{2}{m}\left(K \frac{(N-m)m}{N+1}\right)^2 \geqq k'N, \tag{3.15}$$

where k' is a positive constant. Then by (3.7), (3.8) and (3.15),

$$\frac{1}{N}\sum_{i=1}^{N} h_i^2 \left(\frac{1}{N}\sum_{i=1}^{N} h_i^2\right)^{-k} \leqq Ex^{2k}(k')^{-k} = O(1), \tag{3.16}$$

whence it follows that $EZ_{N1}, \cdots, EZ_{NN}$ satisfies Condition W. By (3.3), (3.4), and (3.5) we have

$$E_0(c_1(R)) = 0,$$

$$V_0(c_1(R)) = E_0(c_1(R))^2 = \frac{1}{N-1}\sum_{i=1}^{N}(d_i - \bar{d})^2 \sum_{i=1}^{N} h_i^2. \tag{3.17}$$

We now obtain from the theorem of Noether:

THEOREM. *If the sequences* $(d_1, \cdots, d_N)$, $N = 1, 2, \cdots$ *satisfy Neother's condition, then under* H_0

$$\frac{c_1(R)}{\sqrt{E_0(c_1(R))^2}} = \frac{\sum_{i=1}^{N} d_i\, EZ_{Nr_i}}{\sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(d_i-\bar{d})^2 \sum_{i=1}^{N}(EZ_{Ni})^2}}$$

has a limiting normal distribution with mean 0 *and variance* 1.

4. Approximate distribution. Pitman [10], [11] has proposed the following approximation of the distribution of a statistic of the type of $c_1(R)$ when the permutations are equally probable. We now assume without loss of generality that $\sum_{i=1}^{N} = d_i = 0$. Let

$$r = r(R) = \frac{\sum_{i=1}^{N} d_i h_{r_i}}{\sqrt{\sum_{i=1}^{N} d_i^2 \sum_{i=1}^{N} h_i^2}} = \frac{c_1(R)}{\sqrt{\sum_{i=1}^{N} d_i^2 \sum_{i=1}^{N} (EZ_{Ni})^2}}.$$

We shall first show that the odd moments of $c_1(R)$, and hence those of r, are zero. If $R = (r_1, \cdots, r_N)$, let $R' = (N+1-r_1, \cdots, N+1-r_N)$. Let $h_i = EZ_{Ni}$, then $-h_i = h_{N+1-i}$; and we have

$$c_1(R) = \sum_{i=1}^{N} d_i h_{r_i} = -\sum_{i=1}^{N} d_i h_{N+1-r_i} = -c_1(R') \quad \text{for all } R.$$

From the above it follows that

$$c_1(R)^{2k+1} = -c_1(R')^{2k+1}. \tag{4.1}$$

Hence

$$E_0[c_1(R)]^{2k+1} = \frac{1}{N!}\sum_R c_1(R)^{2k+1} = -\frac{1}{N!}\sum_R c_1(R')^{2k+1}. \tag{4.2}$$

Since summation over all R is equivalent to summation over all R', we obtain

$$E_0\,[c_1(R)]^{2k+1} = -\frac{1}{N!}\sum_{R'} c_1(R')^{2k+1} = -E_0\,[c_1(R)]^{2k+1}, \tag{4.3}$$

and thus

$$E_0[c_1(R)]^{2k+1} = 0, \qquad E_0(r^{2k+1}) = 0. \tag{4.4}$$

It follows from (3.16) and from Pitman's results that

$$E_0(r^2) = \frac{1}{N-1}, \tag{4.5}$$

$$E_0(r^4) = \frac{3}{(N-1)(N+1)}\left[1 + \frac{(N-2)(N-3)}{3N(N-1)^2}\cdot\frac{k_4}{k_2^2}\cdot\frac{k_4'}{k_2'^2}\right], \tag{4.6}$$

where

$$k_2 = \frac{1}{N-1}\sum_{i=1}^{N} d_i^2$$

$$k_4 = \frac{N}{(N-1)(N-2)(N-3)}\left[(N+1)\sum_{i=1}^{N} d_i^4 - 3\,\frac{N-1}{N}\left(\sum_{i=1}^{N} d_i^2\right)^2\right],$$

and k_2', k_4' are the corresponding functions of the h_i (k_i and k_i' are Fisher's k-statistics).

Since r has the form of a correlation coefficient, we have $-1 \leqq r \leqq 1$, and $E_0(r^{2k+1}) = 0$, $E_0(r^2) = 1/(N-1)$. If the last term in the square brackets of (4.6) is small compared with 1, we also have $E_0(r^4) = 3/[(N-1)(N+1)]$ approximately.

The distribution with the frequency function

$$g(x) = \begin{cases} \dfrac{1}{\mathrm{B}(\frac{1}{2}, \frac{1}{2}N - 1)}\,(1 - x^2)^{\frac{1}{2}N-2}, & -1 \leqq x \leqq 1 \\ 0, & \text{elsewhere} \end{cases} \tag{4.7}$$

has its odd moments equal to zero, and its second and fourth moments, respectively $1/(N-1)$ and $3/[(N-1)\,(N+1)]$.

For this reason the distribution $g(x)$ may be a suitable approximation of the distribution of r under H_0. In Section 7 it is shown that, in the case where the d_i take on two values only (two-sample test), the approximation is satisfactory even for small values of N. Since the distribution $g(x)$ may be a suitable approximation to the distribution of r, we can expect the Student's distribution with $N-2$ degrees of freedom also to approximate the distribution of $(r/\sqrt{1-r^2})\sqrt{N-2}$. Thus we can obtain the approximate critical values of $c_1(R)$ from the t tables.

5. Two-sample statistic $c_1(R)$. Let H_0 be the hypothesis that two samples of m and n observations come from the same continuous population. In this section the rank order test of H_0 will be considered which is most powerful against the alternative that the two samples come from two normal populations with common variance σ^2 and different means $\mu_{1.1}$ and $\mu_{1.2}$ with $\delta = (\mu_{1.2} - \mu_{1.1})/\sigma$ sufficiently small.

The hypothesis H_0 implies that the joint distribution of the $N = m + n$ variables is invariant under all permutations, and that $P(W)$, the probability that all N values are different, is one. Hence the assumptions about H_0 in Section 2 are satisfied. The alternative is a special case of the distribution (2.3) with

$$d_i\xi + \eta = \mu_{1.1}, \qquad i = 1, \cdots, m,$$

$$d_{m+j}\xi + \eta = \mu_{1.2}, \qquad j = 1, \cdots, n.$$

If $d_i = -n/(n + m)$, $i = 1, \cdots, m$; $d_{m+j} = m/(n + m)$, $j = 1, \cdots, n$, then $d = \sum_{i=1}^{N} d_i/N = 0$,

$$\xi = \mu_{1.2} - \mu_{1.1},$$

$$\eta = \frac{m\mu_{1.1} + n\mu_{1.2}}{m + n}, \tag{5.1}$$

$$\delta = \frac{\xi}{\sigma} = \frac{\mu_{1.2} - \mu_{1.1}}{\sigma}.$$

Then, from (2.17),

$$\begin{aligned} c_1(R) &= -\frac{n}{N}\sum_{i=1}^{m} E(Z_{Nr_i}) + \frac{m}{N}\sum_{i=m+1}^{N} E(Z_{Nr_i}) \\ &= -\frac{n}{N}\sum_{i=1}^{N} E(Z_{Nr_i}) + \sum_{i=m+1}^{N} E(Z_{Nr_i}) \\ &= \sum_{j=1}^{n} E(Z_{Nr_{m+j}}). \end{aligned} \tag{5.3}$$

6. Asymptotic distribution of two-sample $c_1(R)$.

THEOREM. *If H_0 is true, the variance of $c_1(R)$ is*

$$V_0(c_1(R)) = \frac{mn}{N(N-1)}\sum_{i=1}^{N} (EZ_{Ni})^2. \tag{6.1}$$

If there exist two constants k, k', $0 < k < k' < 1$, such that $k < n/N < k'$ as $N \to \infty$, then

$$\frac{c_1(R)}{\sqrt{\dfrac{mn}{N(N-1)}\displaystyle\sum_{i=1}^{N} (EZ_{Ni})^2}}$$

has a normal limiting distribution with mean 0 and variance 1.

PROOF. Relation (6.1) follows from (3.16) since

$$\sum_{i=1}^{N} d_{Ni} - \bar{d}_N)^2 = \sum_{i=1}^{N} d_{Ni}^2 = \frac{mn^2}{(m+n)^2} + \frac{nm^2}{(m+n)^2} = \frac{mn}{m+n} = \frac{mn}{N}.$$

By the theorem of Section 2 it now suffices to show that

$$\frac{\sum_{i=1}^{N} d_{Ni}^r}{\left[\sum_{i=1}^{N} d_{Ni}^2\right]^{r/2}} = o(1), \qquad r = 3, 4, \cdots.$$

This follows from the relations

$$\sum_{i=1}^{N} d_{Ni}^2 = \frac{mn}{N} \tag{6.2}$$

and

$$\sum_{i=1}^{N} d_{Ni}^r = m\left(-\frac{n}{N}\right)^r + n\left(\frac{m}{N}\right)^r = N^{-r}[nm^r + (-1)^r mn^r]. \tag{6.3}$$

Hence

$$\frac{\sum_{i=1}^{N} d_{Ni}^r}{\left[\sum_{i=1}^{N} d_{Ni}^2\right]^{r/2}} = n\left(\frac{m}{nN}\right)^{r/2} + (-1)^r m\left(\frac{n}{mN}\right)^{r/2} = o(1), \qquad r = 3, 4, \cdots,$$

since the inequality $0 < k < n/N < k' < 1$ implies $0 < 1 - k' < m/N < 1 - k < 1$.

7. Approximate distribution of two-sample $c_1(R)$. In Section 4 we have considered the approximation of the distribution under H_0 of

$$r = \frac{c_1(R)}{\sqrt{\sum_{i=1}^{N} d_i^2 \sum_{i=1}^{N} (EZ_{Ni})^2}}$$

by the distribution (4.7). In the two-sample case

$$r = \frac{c_1(R)}{\sqrt{\frac{mn}{N} \sum_{i=1}^{N} (EZ_{Ni})^2}},$$

$$E_0(r^2) = \frac{1}{N-1}, \tag{7.1}$$

and

$$E_0(r^4) = \frac{3}{N^2-1}\left[1 + \frac{(N+1)^2}{3N(N-2)(N-3)}\left(\frac{(m-n)^2}{mn} + \frac{6}{N+1}\right)\left(\frac{\mu_4}{\mu_2^2} - \frac{3(N-1)}{N+1}\right)\right].$$

Thus when the second term within the square brackets is small compared with 1, the distribution (4.7) can be expected to approximate the distribution of r.

If X has the distribution $g(x)$, (4.7), then $X\sqrt{N-2}/\sqrt{1-X^2}$ has Student's t-distribution with N-2 degrees of freedom. Hence the latter distribution will approximate the distribution of $r\sqrt{N-2}/\sqrt{1-r^2}$. The t approximation to the critical values of $c_1(R)$ at the level α of the $c_1(R)$ test for given N, n, m, μ_2 is

$$c_{1,\alpha} = \sqrt{\frac{t^2_{N-2,2\alpha}\, mn\mu_2}{t^2_{N-2,2\alpha} + m + n - 2}} \tag{7.2}$$

where $t_{N-2,2\alpha}$ is the tabular value of t with $N - 2$ degrees of freedom at the level 2α in the Fisher and Yates Table III [3].

The t approximation is very good for N as small as 6 insofar as comparisons of fourth moments and critical values are concerned. Shown below are the critical values, $c_{1,\alpha}$, as obtained from the exact distributions for the level of significance $\alpha \cong .05$ and from the t approximation for the level $\alpha = .05$, together with $P\{c\ (R) \geqq c_{1,\alpha}\}$.

A comparison of critical values

N	m	n	$c_{1,\alpha}$ Exact	$c_{1,\alpha}$ t	$P\{c_1(R) \geqq c_{1,\alpha}\}$ Exact	$P\{c_1(R) \geqq c_{1,\alpha}\}$ t
6	3	3	1.81	2.11	.05	.05
7	2	5	1.80	2.11	.05	.05
7	3	4	1.97	2.11	.057	.057
8	3	5	2.12	2.08	.053	.053
8	4	4	2.27	2.15	.057	.057
9	3	6	2.33	2.17	.047	.047
9	4	5	2.42	2.29	.048	.056
10	4	6	2.54	2.40	.048	.052
10	5	5	2.58	2.45	.048	.048

8. Exact distribution of two-sample $c_1(R)$ for small N. The exact distribution of $c_1(R)$ has been computed for small values of N, n, m; $N \leqq 10$ using the Fisher and Yates tables [3]. These distributions are displayed in Table 1, and the values of $c_1(R)$ and U (cf. Sec. 9) are shown for the ${}_NC_n$ distinct permutations of $(x_1, \cdots, x_m, y_1, \cdots, y_n)$, $m + n = N$. Since the probability of each such permutation is $1/{}_NC_n$, these probabilities are not explicitly stated in the table. Since only the relation between x and y matters, we will replace each x_i by a 0 and each y_j by a 1, following the simplifying notation of Wolfowitz [14]. Each R will thus be represented by a sequence of m 0's and n 1's. As the values of $c_1(R)$ for N, m, n and N, n, m are symmetric, the table will show the distribution for subsample sizes m, n; $m \leqq n$. The probability under the null hypothesis that a sample of m 0's and n 1's will give rise to a value of $c_1(R)$ exceeding or equaling the tabulated value of $c_1(R)$ is given in Table 2 for values of $p \leqq 0.10$, and is available for tests of significance when $6 \leqq N \leqq 10$.

TABLE 1

Exact distribution of $C_1(R)$

Exact distribution of $c_1(R)$ for distinct permutations of R, for given N, m, n; $m \leqq n$, together with corresponding values of Mann-Whitney U test [6]. The value of $c_1(R)$ is tabulated under the column heading c_1.

R	c_1	U
$N = 2$	$m = 1$	
01	.56	0
$N = 3$	$m = 1$	
011	.85	0
101	.00	1
$N = 4$	$m = 1$	
0111	1.03	0
1011	.30	1
$N = 4$	$m = 2$	
0011	1.33	0
0101	.73	1
0110	.00	2
$N = 5$	$m = 1$	
01111	1.66	0
10111	.50	1
11011	.00	2
$N = 5$	$m = 2$	
00111	1.66	0
01011	1.16	1
01101	.66	2
10011	.50	2
01110	.00	3
10101	.00	3
$N = 6$	$m = 1$	
011111	1.27	0
101111	.64	1
110111	.20	2
$N = 6$	$m = 2$	
001111	1.91	0
010111	1.47	1
011011	1.07	2
100111	.84	2
011101	.63	3
101011	.44	3
011110	.00	4
101101	.00	4
$N = 6$	$m = 3$	
000111	2.11	0
001011	1.71	1
010011	1.27	2
001101	1.27	2
010101	.83	3
100011	.64	3
001110	.64	3
011001	.43	4
010110	.20	4
100101	.20	4
$N = 7$	$m = 1$	
0111111	1.35	0
1011111	.76	1
1101111	.35	2
1110111	.00	3
$N = 7$	$m = 2$	
0011111	2.11	0
0101111	1.70	1
0110111	1.35	2
1001111	1.11	2
0111011	1.00	3
1010111	.76	3
0111101	.59	4
1011011	.41	4
1100111	.35	4
1101011	.00	5
1011101	.00	5
$N = 7$	$m = 3$	
0001111	2.46	0
0010111	2.11	1
0011011	1.76	2
0100111	1.70	2
0011101	1.35	3
0101011	1.35	3
1000111	1.11	3
0110011	1.00	4
0101101	.94	4
1001011	.76	4
0011110	.76	4
0110101	.59	5
1010011	.41	5
1001101	.35	5
0101110	.35	5
0111001	.24	6
1100011	.00	6
1010101	.00	6
0110110	.00	6
$N = 8$	$m = 2$	
00111111	2.27	0
01011111	1.89	1
01101111	1.57	2
10011111	1.32	2
01110111	1.27	3
10101111	1.00	3
01111011	.95	4
10110111	.70	4
11001111	.62	4
01111101	.57	5
10111011	.38	5
11010111	.32	5
11100111	.00	6
11011011	.00	6
10111101	.00	6
01111110	.00	6
$N = 8$	$m = 3$	
00011111	2.74	0
00101111	2.42	1
00110111	2.12	2
01001111	2.04	2
00111011	1.80	3
01010111	1.74	3
10001111	1.47	3
00111101	1.42	4
01100111	1.42	4
01011011	1.42	4
10010111	1.17	4
01101011	1.10	5
01011101	1.04	5
00111110	.85	5
10011011	.85	5
10100111	.85	5
01110011	.80	6
01101101	.72	6
10101011	.53	6
01011110	.47	6
11000111	.47	6
10011101	.47	6
01110101	.42	7
10110011	.23	7
01101110	.15	7
11001011	.15	7
10101101	.15	7
01111001	.10	8
$N = 8$	$m = 4$	
00001111	2.89	0
00010111	2.59	1
00100111	2.27	2
00011011	2.27	2
00101011	1.95	3
01000111	1.89	3
00011101	1.89	3
00110011	1.65	4
01001011	1.57	4
00101101	1.57	4
10000111	1.32	4
00011110	1.32	4
01010011	1.27	5
00110101	1.27	5
01001101	1.19	5
10001011	1.00	5
00101110	1.00	5
01100011	.95	6
00111001	.95	6
01010101	.89	6
10010011	.70	6
00110110	.70	6
10001101	.62	6
01001110	.62	6
01100101	.57	7
01011001	.57	7
10100011	.38	7
00111010	.38	7
10010101	.32	7

TABLE 1—*Continued*

R	c_1	U	R	c_1	U	R	c_1	U	R	c_1	U
01010110	.32	7	010000011	1.49	5	001011011	2.12	4	001111010	.36	9
			001000101	1.49	5	000111101	2.06	4			
01101001	.25	8	000101001	1.49	5				100110101	.30	9
10001110	.05	7	000100110	1.23	5	010010111	2.06	4	010110101	.30	9
11000011	.00	8				001101011	1.85	5	011100101	.29	10
10100101	.00	8	000110001	1.22	6	010100111	1.79	5	100011110	.28	8
10011001	.00	8	000011010	1.20	5	100001111	1.77	4	110001011	.27	9
			001001001	1.19	6	001011101	1.76	5			
01100110	.00	8	010000101	1.13	6				101001101	.27	9
01011010	.00	8	100000011	.93	6	010011011	1.76	5	011001110	.27	9
00111100	.00	8				001110011	1.58	6	011011001	.26	10
$N = 9$	$m = 2$		001000110	.93	6	000111110	1.50	5	101100011	.09	10
001111111	2.42	0	000101010	.93	6	100010111	1.50	5	100101110	.01	9
010111111	2.06	1	001010001	.92	7	011000111	1.49	6			
011011111	1.76	2	000011100	.84	6				110010011	.00	10
100111111	1.50	2	010001001	.83	7	001101101	1.49	6	001111100	.00	10
011101111	1.49	3				010101011	1.49	6	100111001	.00	10
			000110010	.66	7	010011101	1.40	6	101010101	.00	10
101011111	1.20	3	001100001	.65	8	100100111	1.23	6	0101110101	.00	10
101101111	.93	4	001001010	.63	7	001110101	1.22	7			
011110111	.92	4	100000101	.57	7				10011101	.00	10
011111011	.92	5	000101100	.57	7	010110011	1.22	7	$N = 10$	$m = 2$	
110011111	.84	4				001011110	1.20	6	0011111111	2.54	0
			010000110	.57	7	100011011	1.20	6	0101111111	2.20	1
101110111	.66	5	010010001	.56	8	011001011	1.19	7	0110111111	2.20	2
110101111	.57	5	001010010	.36	8	010101101	1.13	7	1001111111	1.66	2
011111101	.56	6	000110100	.30	8				0111011111	1.66	3
101111011	.36	6	010100001	.29	9	101000111	.93	7	0111101111	1.42	4
110110111	.30	6				001101110	.93	7	1010111111	1.38	3
			100001001	.27	8	100101011	.93	7	0111110111	1.16	5
111001111	.27	6	010001010	.27	8	001111001	.92	8	1011011111	1.12	4
111010111	.00	7	001001100	.27	8	011010011	.92	8	1100111111	1.04	4
110111011	.00	7	001100010	.09	9						
101111101	.00	7	100000110	.01	8	010110101	.86	8	1011101111	.88	5
011111110	.00	7				010011110	.84	7	0111111011	.88	6
$N = 9$	$m = 3$		100010001	.00	9	100011101	.84	7	1101011111	.78	5
000000111	2.99	0	010010010	.00	9	011001101	.83	8	1011110111	.62	6
000001011	2.69	1	001010100	.00	9	001110110	.66	8	1101101111	.54	6
000010011	2.42	2	000111000	.00	9						
000001101	2.33	2	$N = 9$	$m = 4$		100110011	.66	8	0111111101	.54	7
000100011	2.15	3	000011111	3.26	0	011100011	.65	9	1110011111	.50	6
			000101111	2.99	1	101001011	.63	8	1011111011	.34	7
000010101	2.06	3	000110111	2.72	2	110000111	.57	8	1101110111	.28	7
001000011	1.85	4	001001111	2.69	2	100101101	.57	8	1110101111	.26	7
000100101	1.79	4	000111011	2.42	3						
000001110	1.77	3				010101110	.57	8	1111001111	.00	8
000011001	1.76	4	001010111	2.42	3	011010101	.56	9	1110110111	.00	8
			010001111	2.33	3	010111001	.56	9	1101111011	.00	8
000010110	1.50	4	001100111	2.15	4	101010011	.36	9	1011111101	.00	8

TABLE 1—*Continued*

R	c_1	U	R	c_1	U	R	c_1	U	R	c_1	U
0111111110	.00	8	0101111110	.66	8	0110001111	1.92	6	1010011011	.84	9
$N = 10$	$m = 3$		0111011101	.66	9	0010111101	1.92	6	0101110101	.82	10
0001111111	3.20	0	1011100111	.50	9	0100111011	1.92	6	0110101101	.80	10
0010111111	2.92	1	0111110011	.50	10	0011101011	1.76	7			
0011011111	2.66	2	1011011011	.46	9				1100010111	.78	9
0100111111	2.58	2				0101100111	1.70	7	1001011101	.78	9
0011101111	2.42	3	0111101101	.42	10	1001001111	1.66	6	0101011110	.78	9
			1101010111	.40	9	1000110111	1.66	6	1001011110	.78	10
0101011111	2.32	3	1110001111	.38	9	0001111110	1.66	6	1011000111	.62	10
0011110111	2.16	4	1100111011	.38	9	0110010111	1.66	7			
0101101111	2.08	4	1010111101	.38	9				1001110011	.62	10
1000111111	2.04	3				0101011011	1.66	7	0011110110	.62	10
0110011111	2.04	4	0110111110	.38	9	0011011101	1.66	7	0111010011	.62	11
			1011101011	.22	10	0100111101	1.58	7	1010101011	.60	10
0011111011	1.88	5	1101100111	.16	10	0011110011	1.50	8	1100100111	.54	10
0101110111	1.82	5	0111110101	.16	11	0110100111	1.42	8			
0110101111	1.80	5	1001111110	.12	9				1001101101	.54	10
1001011111	1.78	4				0101101011	1.42	8	0101101110	.54	10
1001101111	1.54	5	1110010111	.12	10	0011101101	1.42	8	0111001101	.54	11
			1101011011	.12	10	1001010111	1.40	7	0110110101	.54	11
0111001111	1.54	6	1011011101	.12	10	1010001111	1.38	7	0101111001	.54	11
0110110111	1.54	6	0111011110	.12	10	1000111011	1.38	7			
0101111011	1.54	6	1100111101	.04	10				1000111110	.50	9
0011111101	1.54	6	$N = 10$	$m = 4$		0010111110	1.38	7	1100011011	.50	10
1010011111	1.50	5	0000111111	3.58	0	0110011011	1.38	8	1010011101	.50	10
			0001011111	3.32	1	0101011101	1.32	8	0110011110	.50	10
1001110111	1.28	6	0001101111	3.08	2	1001100111	1.16	8	0111100011	.38	12
0111010111	1.28	7	0010011111	3.04	2	0111000111	1.16	9			
1010101111	1.26	6	0001110111	2.82	3				1011001011	.34	11
0110111011	1.26	7				0101110011	1.16	9	1010110011	.34	11
0101111101	1.20	7	0010101111	2.80	3	0011110101	1.16	9	0011111010	.34	11
			0100011111	2.70	3	0110101011	1.14	9	1101000111	.28	11
1100011111	1.16	6	0011001111	2.54	4	1010010111	1.12	8	1001110101	.28	11
0111100111	1.04	8	0010110111	2.54	4	1001011011	1.12	8			
1011001111	1.00	7	0001111011	2.54	4				0101110110	.28	11
1010110111	1.00	7				0011011110	1.12	8	0111010101	.28	12
1001111011	1.00	7	0100101111	2.46	4	0101101101	1.08	9	1100101011	.26	11
			0011010111	2.28	5	1100001111	1.04	8	1010101101	.26	11
0011111110	1.00	7	0010111011	2.26	5	1000111101	1.04	8	0110101110	.26	11
0111011011	1.00	8	0101001111	2.20	5	0100111110	1.04	8			
1100101111	.92	7	0001111101	2.20	5				0110111001	.26	12
0110111101	.92	8				0110011101	1.04	9	1100011101	.16	11
0111101011	.76	9	0100110111	2.20	5	1010100111	.88	9	1011010011	.08	12
			1000011111	2.16	4	1001101011	.88	9	0111100101	.04	13
1011010111	.74	8	0011100111	2.04	6	0011101110	.88	9	1001101110	.00	11
1010111011	.72	8	0011011011	2.00	6	0111001011	.88	10			
1101001111	.66	8	0101010111	1.94	6				1110000111	.00	12
1100110111	.66	8				0110110011	.88	10	1101001011	.00	12
1001111101	.66	8	1000101111	1.92	5	0011111001	.88	10	1100110011	.00	12

TABLE 1—*Continued*

R	c_1	U	R	c_1	U	R	c_1	U	R	c_1	U
1011001101	.00	12	0010011101	2.04	6	0100011110	1.16	8	0011100110	.50	11
1010110101	.00	12	0011001011	1.88	7				0011110001	.50	12
			0010110011	1.88	7	0101100011	1.04	10	0111000011	.50	12
1001111001	.00	12	0101000111	1.82	7	0011100101	1.04	10			
0111001110	.00	12	0001110101	1.82	7	1010000111	1.00	9	1010010011	.46	11
0110110110	.00	12				1000110011	1.00	9	0011011010	.46	11
0101111010	.00	12	0100101011	1.80	7	1001001011	1.00	9	0110100101	.42	12
0011111100	.00	12	0010101101	1.80	7				0101101001	.42	12
			1000010111	1.78	6	0011001110	1.00	9	1001010101	.40	11
0111011001	.00	13	0001011110	1.78	6	0010110110	1.00	9			
$N = 10$	$m = 5$		0100011101	1.70	7	0001111010	1.00	9	0101010110	.40	11
0000011111	3.70	0				0110010011	1.00	10	1000101110	.38	10
0000101111	3.46	1	0011010011	1.62	8	0011011001	1.00	10	1100001011	.38	11
0000110111	3.20	2	1000100111	1.54	7				1010001101	.38	11
0001001111	3.20	2	0001101110	1.54	7	0101010101	.94	10	1000111001	.38	11
0001010111	2.94	3	0110000111	1.54	8	0100101110	.92	9			
			0101001011	1.54	8	1000101101	.92	9	0110001110	.38	11
0010001111	2.92	3				0110001101	.92	10	0100111010	.38	11
0000111011	2.92	3	0100110011	1.54	8	0100111001	.92	10	0010111100	.38	11
0001100111	2.70	4	0011001101	1.54	8				0110011001	.38	12
0010010111	2.66	4	0010110101	1.54	8	0110100011	.76	11	1010100011	.22	12
0001011011	2.66	4	0001111001	1.54	8	0011101001	.76	11			
			1000011011	1.50	7	1001010011	.74	10	0011101010	.22	12
0000111101	2.58	4				0011010110	.74	10	1001100101	.16	12
0100001111	2.58	4	0010011110	1.50	7	1010001011	.72	10	0101100110	.16	12
0010100111	2.42	5	0100101101	1.46	8				0111000101	.16	13
0001101011	2.42	5	0011100011	1.38	9	0010111010	.72	10	0101110001	.16	13
0010011011	2.38	5	1001000111	1.28	8	0101100101	.70	11			
			0001110110	1.28	8	1001001101	.66	10	0110101001	.14	13
0100010111	2.32	5				1000110101	.66	10	1001001110	.12	11
0001011101	2.32	5	0101010011	1.28	9	1100000111	.66	10	1000110110	.12	11
0011000111	2.16	6	0011010101	1.28	9				1100010011	.12	12
0001110011	2.16	6	1000101011	1.26	8	0101001110	.66	10	1010010101	.12	12
0010101011	2.14	6	0010101110	1.26	8	0100110110	.66	10			
			0010111001	1.26	9	0001111100	.66	10	0110010110	.12	12
0100100111	2.08	6				0110010101	.66	11	0101011010	.12	12
0001101101	2.08	6	0110001011	1.26	9	0101011001	.66	11	0011011100	.12	12
0000111110	2.04	5	0101001101	1.20	9				1001011001	.12	12
1000001111	2.04	5	0100110101	1.20	9	1000011110	.62	9	1100001101	.04	12
0100011011	2.04	6	1000011101	1.16	8	1001100011	.50	11	0100111100	.04	12

9. The $c_1(R)$ test compared with Mann and Whitney U test. The two best known rank order tests now in current use for testing whether two samples come from the same population are the Mann and Whitney U test [6] and the Wilcoxon T test [16]. These two tests have been developed to test the null hypothesis that the sample $(x_1, \cdots, x_m)$ comes from the same population as the sample $(y_1, \cdots, y_n)$ with cdf $F(\xi)$ against the alternative that the cdf of $(y_1, \cdots, y_n)$,

TABLE 2

Critical values of $c_1(R)$

Listed below are the critical values of $c_1(R)$ tabulated as c_1 for the two-sample case for values of N, m such that $6 \leqq N \leqq 10$, $1 < m \leqq [N/2]$, together with the probability (p) of exceeding this value on the null hypothesis, where $p \leqq .10$.

N	m	c_1	p
6	3	2.11	.050
		1.71	.100
7	2	2.11	.048
		1.70	.095
7	3	2.46	.029
		2.11	.057
		1.76	.086
8	2	2.27	.036
		1.89	.071
8	3	2.75	.018
		2.42	.036
		2.12	.054
		2.04	.071
		1.80	.089
8	4	2.89	.014
		2.59	.029
		2.27	.057
		1.95	.071
		1.89	.100
9	2	2.42	.028
		2.06	.056
		1.76	.083
9	3	2.99	.012
		2.69	.023
		2.42	.035
		2.33	.048
		2.06	.071
		1.85	.083
		1.79	.095
9	4	3.26	.008
		2.99	.016
		2.69	.032
		2.42	.048
		2.33	.056
		2.15	.063
		2.12	.071
		2.06	.087
		1.85	.095
10	2	2.54	.022
		2.20	.067
10	3	3.20	.008
		2.92	.016
		2.66	.025
		2.58	.033
		2.42	.041
		2.32	.050
		2.16	.058
		2.08	.067
		2.04	.083
		1.88	.091
		1.82	.100
10	4	3.32	.010
		3.04	.019
		2.80	.029
		2.70	.033
		2.54	.048
		2.28	.057
		2.20	.076
		2.04	.086
		1.94	.095
10	5	3.46	.008
		2.92	.028
		2.66	.040
		2.58	.048
		2.32	.067
		2.14	.079
		2.08	.087

$G(\xi)$ have the relation $F(\xi) > G(\xi)$ for every ξ. The U test has been shown to be equivalent to the Wilcoxon T test [6]. In fact Mann and Whitney show the following linear relation between the U statistic and the T statistic:

$$U = mn + \tfrac{1}{2}(n)(n+1) - T. \tag{9.1}$$

The U test has been defined by Mann and Whitney as follows. In an ordered sample of m x's and n y's, let U count the number of times a y precedes an x. If $P(U < \bar{U})$ equals α under the null hypothesis, the test will be considered significant at the significance level α if $U < \bar{U}$ and the hypothesis of identical distribution of x and y will be rejected. Wilcoxon's T is the sum of the ranks in the sample of n observations.

The $c_1(R)$ test is generally more sensitive than the U test or the equivalent T test, for in many permutations where $U(R_i) = U(R_j)$, $c_1(R_i) \neq c_1(R_j)$, $i \neq j$, with the result that although

$$P\{U \leqq U(R_i)\} = P\{U \leqq U(R_j)\},$$
$$P\{c_1(R) \geqq c_1(R_i)\} \neq P\{c_1(R) \geqq c_1(R_j)\}, \qquad i \neq j.$$

The most powerful test is not difficult to compute, as it involves at most n additions or subtractions of numbers readily accessible in the Fisher and Yates tables [3]. Table XX tabulates for $N \leqq 50$ the positive values of EZ_{Ni}.

There is a very close dependence between these two statistics, as a scrutiny of Table 1 will show. For all $N \leqq 8$, $m \geqq 1$, U is a nonincreasing function of c_1, and for all N, $m = 1$, U is a strictly decreasing function of c_1. In general, however, there is no such functional relationship. We shall now derive the correlation under H_0 between the U and the c_1 statistic.

By definition [3], the correlation $\rho[c_1(R), U]$ can be written

$$\rho[c_1(R), U] = \frac{E_0[c_1(R)U]}{\sqrt{V_0(c_1(R)V(U))}}. \tag{9.2}$$

Since there is a linear relation [6] between the Mann and Whitney U and Wilcoxon's T, to wit: $U = mn + \frac{1}{2}m(n+1) - T$, it follows that

$$V_0(U) = V_0(T), \qquad E_0(c_1(R)U) = -E_0(c_1(R)T), \tag{9.3}$$

where T is the sum of the ranks of the sample of n observations. We shall use Wilcoxon's statistic to evaluate $E_0(c_1(R)U)$. Now

$$E_0(c_1(R)T) = \frac{1}{{}_NC_n} \sum \left(\sum_{i=m+1}^{m+n} EZ_{Nr_i} \right) \left(\sum_{j=m+1}^{m+n} r_j \right), \tag{9.4}$$

where $\sum$ denotes the summation over the ${}_NC_n$ distinct combinations. Consider an $r_j = k$ fixed. Then the coefficients, EZ_{Nr_i}, of k in the summation occur ${}_{N-1}C_{n-1}$ times if $r_i = k$, and ${}_{N-2}C_{n-2}$ times if $r_i \neq k$. Since $\sum_{i=1}^{N} kEZ_{Nr_i} = 0$, the coefficient of k reduces to $({}_{N-1}C_{n-1} - {}_{N-2}C_{n-2})EZ_{Nk}$. Hence it follows that

$$\begin{aligned} E(c_1(R)T) &= \frac{1}{{}_NC_n} ({}_{N-1}C_{n-1} - {}_{N-2}C_{n-2}) \sum_{k=1}^{N} kE(Z_{nk}) \\ &= \frac{mn}{N(N-1)} \sum_{k=1}^{N} kEZ_{Nk}. \end{aligned} \tag{9.5}$$

Now

$$\sum_{i=1}^{N} iEZ_{Ni} = \sum_{i=1}^{N} iN \binom{N-1}{i-1} \int_{-\infty}^{\infty} xf(x)[F(x)]^{i-1}[1-F(x)]^{N-1}\,dx \tag{9.6}$$
$$= N(N-1) \int_{-\infty}^{\infty} xf(x)F(x)\,dx,$$

after interchanging the order of summation and integration and using the well known properties of a binomial. Integrating the right-hand side by parts we have

$$\sum_{i=1}^{N} iEZ_{Ni} = N(N-1) \int_{-\infty}^{\infty} [f(x)]^2\,dx = \frac{N(N-1)}{2\sqrt{\pi}}. \tag{9.7}$$

Now

$$V_0(U) = V_0(T) = \frac{mn(N+1)}{12}, \tag{9.8}$$

[6] and

$$V_0(c_1(R)) = \frac{mn\mu_2}{N-1}, \tag{9.9}$$

where $\mu_2 = \sum (EZ_{Ni})^2/N$. Since $E_0c_1(R) = 0$, the covariance of $c_1(R)$ and T is equal to $E_0(c_1(R)T)$. Hence

$$\rho = -\frac{\dfrac{mn}{2\sqrt{\pi}}}{mn\sqrt{\dfrac{N+1}{12(N-1)}\mu_2}} = -\sqrt{\frac{3(N-1)}{\pi(N+1)\mu_2}}. \tag{9.10}$$

It has been shown in an unpublished paper of W. Hoeffding that

$$\lim_{N\to\infty} \sum_{i=1}^{N} (EZ_{Ni})^2/N = 1, \text{ and hence } \lim_{N\to\infty} \rho = -\sqrt{3/\pi} \cong -0.9772050.$$

Thus there is not a linear functional relationship between these two statistics. We do, however, note that, for any given N, ρ is independent of the subdivision of N into groups of m and n, $m + n = N$.

Numerical values of ρ can then easily be computed using Tables XX and XXI of Fisher and Yates [3]. From numerical calculation, $|\rho| > .99$, $N \leqq 15$.

10. The power of the $c_1(R)$ test. Since no method has yet been found for evaluating analytically the power function of the $c_1(R)$ test, the power of the $c_1(R)$ test has been investigated by drawing two random groups of samples of size $N = 8$, $m = 4$, $n = 4$ for $\delta = 0, .1, .2, .3, .4, .5, .6, .7, .8, .9, 1.0, 1.2, 1.5, 2, 2.5, 3, 4$. The samples were drawn by selecting 100 groups of eight pairs of digits except the pair 00 from the Fisher and Yates tables of random numbers [3]. These pairs were uniformly distributed over the range 1–99. These 800 pairs of two-digit numbers were then transformed into standard normal deviates x

using the relationship $F(x) = \int_{-\infty}^{x} f(x)\, dx$, where $F(x) = (.01)\ (N_i)$, $N_i = 1, \cdots, 99$.

The first four pairs in each group became the y sample and the last four pairs became the x sample. Then a constant, δ, was added to each value of y and the $c_1(R)$ test carried out. The results were plotted as number of times the H_a (Section 5) was accepted per 100 against δ, and a smooth curve fitted to the pairs of points.

Neyman and Tokarska [8] give tables of the power function of the one-tailed Student's t test for levels of significance $\alpha = .05$ and .01. Although the sampling procedure yields the approximate power function for $\alpha = .057$, and the Neyman and Tokarska power function is given for $\alpha = .05$, general comparisons can be made on the evidence of shape. No attempt is made to compare actual values.

We can place confidence limits on $\beta(\delta)$, the actual value of the power, and evaluate the reliability of the sampling estimates of $\beta(\delta)$. Let M be the total number of observations and s the number of observations that lead to rejection. Then s has a binomial distribution with parameters M and $\beta(\delta)$, and $E(s) = M\beta((\delta)$; $V(S) = M\beta(\delta)(1 - \beta(\delta))$.

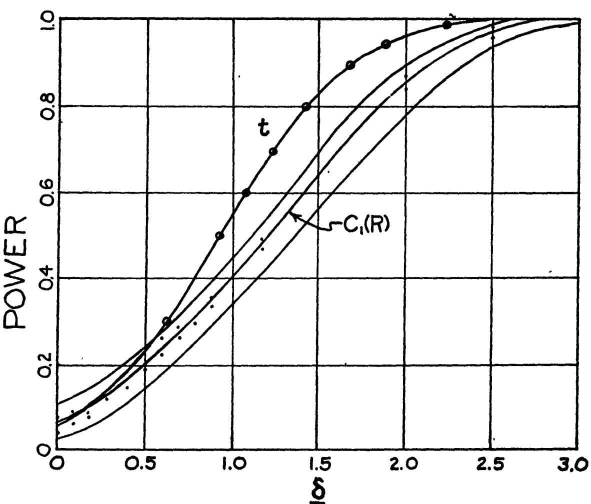

FIG. 1

For large M, $(s - M\beta(\delta))/\sqrt{\beta(\delta)(1 - \beta(\delta))M} \sim M(0, 1)$ and we have as confidence limits on $\beta(\delta)$ at the level α [1]

$$\frac{s + \frac{1}{2}t_\alpha^2 - t_\alpha \sqrt{s\left(1 - \frac{s}{M}\right) + \frac{1}{4}t_\alpha^2}}{M + t_\alpha^2} \tag{10.1}$$

$$\leqq \beta(\delta) \leqq \frac{s + \frac{1}{2}t_\alpha^2 + t_\alpha \sqrt{s\left(1 - \frac{s}{M}\right) + \frac{1}{4}t_\alpha^2}}{M + t_\alpha^2}.$$

Figure 1 shows the power curve of the one-tailed t-test and that of the c_1 test; 95% confidence bands for the true power of the c_1 test are shown.

Curve t is the power curve of the one-tailed t-test with 6 d.f. such that $\beta(0) = .05$. Curve $c_1(R)$ is the power curve of the $c_1(R)$ test with $N = 8$, $m = 4$, $n = 4$ at the .057 level, together with the confidence band on the true power of the $c_1(R)$.

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