EXTENSION OF A METHOD OF INVESTIGATING THE PROPERTIES OF ANALYSIS OF VARIANCE TESTS TO THE CASE OF RANDOM AND MIXED MODELS

By F. N. David and N. L. Johnson

University College, London

Summary. Results are given whereby the methods described in an earlier paper [1], dealing with the parametric case, may be applied also to the case of random, or mixed random and parametric components.

1. Introduction. In a recent paper [1] we set out a method for approximating to the power function of tests of the general linear hypothesis under fairly wide conditions of non-normality and non-uniformity of residual variance. In many analysis of variance problems, it is more reasonable to replace some or all of the parameters by independent random variables with zero expected value. (This is the basis of the well-known 'components of variance' model.)

In the present paper we give certain general formulae which will facilitate the application of the method described in [1] to such random or mixed models. Our results are presented in such a form that they refer to the various sums of squares suggested by the analysis appropriate to the parametric case. Since, however, the same sums of squares are commonly used (though not necessarily in the same way) in the analysis when a random or mixed model is envisaged, the results given will be appropriate in such cases, though care must be taken in their interpretation.

It may be noted that this extension of our method covers the case of the general linear hypothesis with correlated residuals, since such residuals may be represented as the sum of

(i) independent residuals for each observation, and
(ii) independent random terms common to different observations (i.e., occurring in the same way as do parameters in the general linear model).

2. The theoretical model. In [1] we used a theoretical model of the form

$$x_i = a_{i1}\theta_1 + \cdots + a_{i,s-p}\theta_{s-p} + a_{i,s-p+1}\theta_{s-p+1} + \cdots + a_{is}\theta_s + z_i \qquad (i = 1, \cdots, n),$$

where the θ's were unknown parameters and the z's were independent random variables each with zero expected value. The hypothesis to be tested specified that $\theta_{s-p+1} = \cdots = \theta_s = 0$.

We now replace $\theta_{q+1}, \cdots, \theta_{s-p}, \theta_{s-p+r+1}, \cdots, \theta_s (q < s - p, r < p)$ by independent random variables $y_{q+1}, \cdots, y_{s-p}, y_{s-p+r+1}, \cdots, y_s$ (each with expected value zero) so that the theoretical model is of form

$$\begin{aligned} x_i = a_{i1}\theta_1 + \cdots + a_{iq}\theta_q + a_{i,q+1}y_{q+1} + \cdots + a_{i,s-p}y_{s-p} \\ + a_{i,s-p+1}\theta_{s-p+1} + \cdots + a_{i,s-p+r}\theta_{s-p+r} \\ + a_{i,s-p+r+1}y_{s-p+r+1} + \cdots + a_{is}y_s + z_i . \end{aligned}$$

The hypothesis to be tested specifies

$$\theta_{s-p+1} = \cdots = \theta_{s-p+r} = 0,$$

$$\sigma(y_{s-p+r+1}) = \cdots = \sigma(y_s) = 0.$$

As in [1] it is also assumed that the matrix $A = (a_{ij})$ is nonsingular and the z's are mutually independent. We further assume that the y's are independent of the z's.

3. Method of investigation. It will be recalled that in the parametric case the test of the hypothesis $H(\theta_{s-p+1} = \cdots = \theta_s = 0)$ was based on the criterion $(S_b/p)/(S_a/(n - s))$, where S_a is the minimum value of

$$\sum_{i=1}^{n} (x_i - a_{i1}\theta_1 - \cdots - a_{is}\theta_s)^2$$

with respect to $\theta_1, \cdots, \theta_s$; and $S_a + S_b$ is the minimum value of

$$\sum_{i=1}^{n} (x_i - a_{i1}\theta_1 - \cdots - a_{i,s-p}\theta_{s-p})^2$$

with respect to $\theta_1, \cdots, \theta_{s-p}$. The upper $100\alpha\%$ limit of the test criterion could be obtained from tables of significance limits of the F-distribution. The test could formally be expressed as

$$\text{reject } H \text{ if } (S_b/p)/(S_a/(n - s)) > F_{p,n-s,\alpha}.$$

Investigation of properties of the test reduces to evaluation of the probability

$$P\{(S_b/p)/(S_a/(n - s)) > F_{p,n-s,\alpha}\}$$

which can be written in the form

$$P\{S_r - CS_a > 0\},$$

where $C = 1 + pF_{p,n-s,\alpha}/(n - s)$ and $S_r = S_a + S_b$. This probability is obtained approximately by finding a frequency curve which has the same first four moments as $S_r - CS_a$. It is assumed that the theoretical model (1) is adequate in the number of parameters and/or random variables which it contains. Following our previous work it may be shown that S_a and S_r may be written in canonical form as

$$S_a = \sum_{i,j=1}^{n} m_{ij}\, z_i z_j$$

and

$$S_r = \sum_{i,j=1}^{n} m'_{ij}\, (z_i + D'_i)(z_j + D'_j),$$

where

$$D'_i = \sum_{t=s-p+1}^{s-p+r} a_{it}\,\theta_t + \sum_{t=s-p+r+1}^{s} a_{it}\, y_t$$

$$= A_i + Y_i \qquad (i = 1, \cdots, n)$$

and the m's and m''s depend only on the a's (see Section 4).

4. Definition of determinants. Before turning to a consideration of the moments it will be convenient to summarise in determinantal form the various quantities which are required.

As before

$$G_{jk} = \sum_{i=1}^{n} a_{ij} a_{ik}$$

and

$$\Delta = \begin{vmatrix} G_{11} & \cdots & G_{1s} \\ \vdots & & \vdots \\ G_{1s} & \cdots & G_{ss} \end{vmatrix}, \qquad \Delta' = \begin{vmatrix} G_{11} & \cdots & G_{1,s-p} \\ \vdots & & \vdots \\ G_{1,s-p} & \cdots & G_{s-p,s-p} \end{vmatrix}.$$

Let

$$\Delta'_{ii} = \begin{vmatrix} 1 & a_{i1} & \cdots & a_{i,s-p} \\ a_{i1} & G_{11} & \cdots & G_{1,s-p} \\ \vdots & \vdots & & \vdots \\ a_{i,s-p} & G_{1,s-p} & \cdots & G_{s-p,s-p} \end{vmatrix}, \qquad \alpha'_{ij} = \begin{vmatrix} 0 & a_{i1} & \cdots & a_{i,s-p} \\ a_{j1} & G_{11} & \cdots & G_{1,s-p} \\ \vdots & \vdots & & \vdots \\ a_{j,s-p} & G_{1,s-p} & \cdots & G_{s-p,s-p} \end{vmatrix}.$$

Then

$$m'_{ij} = -\alpha'_{ij}/\Delta' \qquad i \neq j;$$

$$m'_{ii} = 1 - \alpha'_{ii}/\Delta' = \Delta'_{ii}/\Delta'.$$

Similar quantities without primes may be expressed as similar determinants of order $(s + 1)$ instead of $(s - p + 1)$. In this present work we shall also use

$$\delta'_i = \frac{1}{\Delta'} \begin{vmatrix} 0 & \sum_i a_{i1} A_i & \cdots & \sum_i a_{i,s-p} A_i \\ a_{i1} & G_{11} & \cdots & G_{1,s-p} \\ \vdots & \vdots & & \vdots \\ a_{i,s-p} & G_{1,s-p} & \cdots & G_{s-p,s-p} \end{vmatrix} = \sum_{j=1}^{n} m'_{ij} A_i,$$

$$\Delta'_A = \frac{1}{\Delta'} \begin{vmatrix} \sum_i A_i^2 & \sum_i a_{i1} A_i & \cdots & \sum_i a_{i,s-p} A_i \\ \sum_i a_{i1} A_i & G_{11} & \cdots & G_{1,s-p} \\ \vdots & \vdots & & \vdots \\ \sum_i a_{i,s-p} A_i & G_{1,s-p} & \cdots & G_{s-p,s-p} \end{vmatrix} = \sum_{i,j=1}^{n} m'_{ij} A_i A_j.$$

Similar quantities without primes which may be expressed as determinants of order $(s + 1)$ will have zero value. So far the determinants are the same or are directly comparable with those of our previous paper. We now introduce new determinants and note in these definitions that t and u may run from $(s - p + r + 1)$ to s only. We define

$$\Gamma'_{tu} = \frac{1}{\Delta'}\begin{vmatrix} G_{tu} & G_{1t} & \cdots & G_{s-p,t} \\ G_{1u} & G_{11} & \cdots & G_{1,s-p} \\ \vdots & \vdots & & \vdots \\ G_{s-p,u} & G_{1,s-p} & \cdots & G_{s-p,s-p} \end{vmatrix}, \qquad \Omega'_{it} = \frac{1}{\Delta'}\begin{vmatrix} a_{it} & a_{i1} & \cdots & a_{i,s-p} \\ G_{1t} & G_{11} & \cdots & G_{1,s-p} \\ \vdots & \vdots & & \vdots \\ G_{s-p,t} & G_{1,s-p} & \cdots & G_{s-p,s-p} \end{vmatrix},$$

$$\Lambda'_t = \frac{1}{\Delta'}\begin{vmatrix} \sum_i a_{it} A_i & \sum_i a_{i1} A_i & \cdots & \sum_i a_{i,s-p} A_i \\ G_{1t} & G_{11} & \cdots & G_{1,s-p} \\ \vdots & \vdots & & \vdots \\ G_{s-p,t} & G_{1,s-p} & \cdots & G_{s-p,s-p} \end{vmatrix}.$$

Similar determinants of order $(s + 1)$ may be written down to represent quantities without primes but these will be zero.

5. Moments of S_r and S_a. We write

$$\mu(S_r^l S_a^m) = \mathcal{E}[(S_r - \mathcal{E}(S_r))^l (S_a - \mathcal{E}(S_a))^m]$$

with $\kappa(S_r^l S_a^m)$ for the corresponding cumulants. It is easy to see that the moments of S_a are the same as those indicated in [1] with the appropriate determinants now put equal to zero. For example, (all summations running from 1 to n)

$$\mathcal{E}(S_a) = \sum_i m_{ii}\,\kappa_{2i},$$

$$\kappa(S_a^2) = \sum_i m_{ii}^2\,\kappa_{4i} + 2\sum_i\sum_j m_{ij}\,\kappa_{2i}\,\kappa_{2j},$$

$$\kappa(S_a^3) = \sum_i m_{ii}^3\,\kappa_{6i} + 12\sum_i\sum_j m_{ii}m_{ij}^2\,\kappa_{4i}\,\kappa_{2j} + 6\sum_i\sum_j m_{ii}m_{ij}m_{jj}\,\kappa_{3i}\,\kappa_{3j} + 4\sum_i\sum_j m_{ij}^3\,\kappa_{3i}\,\kappa_{3j} + 8\sum_i\sum_j\sum_l m_{ij}m_{il}m_{jl}\,\kappa_{2i}\,\kappa_{2j}\,\kappa_{2l},$$

and so on, the rth cumulant of z_i being defined as κ_{ri} for $r \geqq 2$. Again it is a simple matter to show that $\kappa(S_r S_a^l)$ is the same under this treatment as it was in [1] if the appropriate changes are made in the determinants. Thus

$$\kappa(S_r S_a) = \sum_i m_{ii}m'_{ii}\,\kappa_{4i} + 2\sum_i\sum_j m_{ij}m'_{ij}\,\kappa_{2i}\,\kappa_{2j} + 2\sum_i m_{ii}\,\delta'_i\,\kappa_{3i},$$

where δ'_i has A's instead of D's in its definition. The moments of S_r and the cross cumulants of S_a and S_r containing a power of S_r greater than or equal to 2 can be derived by elementary algebra or by a simple combinatorial method from the moments of S_r previously obtained. Let $\bar{\kappa}_{rt}$ be the rth cumulant of y_t.

We have then

$$\mathcal{E}(S_r) = \sum_i m'_{ii}\kappa_{2i} + \Delta'_A + \sum_t \Gamma'_{tt}\bar{\kappa}_{2t},$$

where t may run from $(s - p + r + 1)$ to s only. Again

$$\kappa(S_r^2) = \sum_i m'^2_{ii}\kappa_{4i} + 2\sum_i\sum_j m'^2_{ij}\kappa_{2i}\kappa_{2j} + 4\sum_i m'_{ii}\delta'_i\kappa_{2i} + 4\sum_i \delta'^2_i\kappa_{2i}$$
$$+ 4\sum_t \Gamma'^2_{tt}\bar{\kappa}_{4t} + 2\sum_t\sum_u \Gamma'^2_{tu}\bar{\kappa}_{2t}\bar{\kappa}_{2u} + 4\sum_t \Gamma'_{tt}\Lambda'_t\bar{\kappa}_{3t}$$
$$+ 4\sum_t \Lambda'^2_t\bar{\kappa}_{2t} + 4\sum_i\sum_t \Omega'^2_{it}\bar{\kappa}_{2t}\kappa_{2i}$$

This last expression demonstrates how the moments of S_r can be obtained directly by substitution from [1]. We write down the expression for $\kappa(S_r^2)$ from [1] and add to it expressions in t, or in t and u, which we obtain by substituting $\bar{\kappa}_{rt}$ for κ_{ri}, Γ'_{tt} for m'_{ii}, and so on. We add further the terms in $\bar{\kappa}_{rt}\kappa_{ri}$ by making the appropriate substitution for the cumulants and writing Ω'_{it} for m'_{ij}. This combinatorial method is obvious if the form of the various determinants is considered. We have worked out the cumulants and cross-cumulants up to and including those of the fourth order by two different methods but they are so easily derived by the above process that we do not reproduce them here in full generality.

6. Special cases of normality. If it is assumed that z_i and y_t are both normally distributed then from a knowledge of the moments it is possible to study the effect of heterogeneity of variance on the power function of the test. For the special case of normality we have

$$\mathcal{E}(S_a) = \sum m_{ii}\kappa_{2i},$$
$$\mathcal{E}(S_r) = \sum m'_{ii}\kappa_{2i} + \Delta'_A + \sum \Gamma'_{tt}\bar{\kappa}_{2t},$$
$$\kappa(S_a^2) = 2\sum m^2_{ij}\kappa_{2i}\kappa_{2j},$$
$$\kappa(S_aS_r) = 2\sum m_{ij}m'_{ij}\kappa_{2i}\kappa_{2j},$$
$$\kappa(S_r^2) = 2\sum m'^2_{ij}\kappa_{2i}\kappa_{2j} + 4\sum \delta'^2_i\kappa_{2i} + 2\sum \Gamma'^2_{tu}\bar{\kappa}_{2t}\bar{\kappa}_{2u} + 4\sum \Lambda'^2_t\bar{\kappa}_{2t}$$
$$\kappa(S_a^3) = 8\sum m_{ij}m_{il}m_{jl}\kappa_{2i}\kappa_{2j}\kappa_{2l},$$
$$\kappa(S_rS_a^2) = 8\sum m_{ij}m_{il}m'_{jl}\kappa_{2i}\kappa_{2j}\kappa_{2l},$$
$$\kappa(S_r^2S_a) = 8\sum m'_{ij}m'_{il}m_{jl}\kappa_{2i}\kappa_{2j}\kappa_{2l} + 8\sum m_{ij}\delta'_i\delta'_j\kappa_{2i}\kappa_{2j} + 8\sum m_{ij}\Omega'_{it}\Omega'_{jt}\kappa_{2i}\kappa_{2j}\bar{\kappa}_{2t},$$
$$\kappa(S_r^3) = 8\sum m'_{ij}m'_{il}m'_{jl}\kappa_{2i}\kappa_{2j}\kappa_{2l} + 8\sum \Gamma'_{tu}\Gamma'_{tv}\Gamma'_{uv}\bar{\kappa}_{2t}\bar{\kappa}_{2u}\bar{\kappa}_{2v}$$
$$+ 24\sum m'_{ij}\Omega'_{it}\Omega'_{jt}\kappa_{2i}\kappa_{2j}\bar{\kappa}_{2t} + 24\sum \Omega'_{it}\Omega'_{iu}\Gamma'_{tu}\kappa_{2i}\bar{\kappa}_{2t}\bar{\kappa}_{2u} + 24\sum m'_{ij}\delta'_i\delta'_j\kappa_{2i}\kappa_{2j}$$
$$+ 48\sum \Omega'_{it}\delta'_i\Lambda'_t\kappa_{2i}\bar{\kappa}_{2t} + 24\sum \Gamma'_{tu}\Lambda'_t\Lambda'_u\bar{\kappa}_{2t}\bar{\kappa}_{2u},$$

$$\kappa(S_a^4) = 48 \sum m_{ij} m_{il} m_{jk} m_{lk} \kappa_{2i} \kappa_{2j} \kappa_{2l} \kappa_{2k} ,$$

$$\kappa(S_r S_a^3) = 48 \sum m'_{ij} m_{il} m_{jk} m_{lk} \kappa_{2i} \kappa_{2j} \kappa_{2l} \kappa_{2k} ,$$

$$\kappa(S_r^2 S_a^2) = 48 \sum m'_{ij} m'_{il} m_{jk} m_{lk} \kappa_{2i} \kappa_{2j} \kappa_{2l} \kappa_{2k} + 48 \sum m_{ij} m_{il} \Omega'_{jt} \Omega'_{lt} \kappa_{2i} \kappa_{2j} \kappa_{2l} \bar{\kappa}_{2t} + 32 \sum m_{ij} m_{il} \delta'_j \delta'_l \kappa_{2i} \kappa_{2j} \kappa_{2l} ,$$

$$\kappa(S_r^3 S_a) = 48 \sum m'_{ij} m'_{il} m'_{jk} m_{lk} \kappa_{2i} \kappa_{2j} \kappa_{2l} \kappa_{2k} + 96 \sum m'_{ij} m_{il} \Omega'_{jt} \Omega'_{lt} \kappa_{2i} \kappa_{2j} \kappa_{2l} \bar{\kappa}_{2t} + 48 \sum m_{ij} \Omega'_{it} \Omega'_{ju} \Gamma'_{tu} \kappa_{2i} \kappa_{2j} \bar{\kappa}_{2t} \bar{\kappa}_{2u} + 96 \sum m'_{ij} m_{il} \delta'_j \delta'_l \kappa_{2i} \kappa_{2j} \kappa_{2l} + 96 \sum m_{ij} \delta'_j \Omega'_{it} \Lambda'_t \kappa_{2i} \kappa_{2j} \bar{\kappa}_{2t} ,$$

$$\begin{aligned}\kappa(S_r^4) = {} & 48 \sum m'_{ij} m'_{il} m'_{jk} m'_{lk} \kappa_{2i} \kappa_{2j} \kappa_{2l} \kappa_{2k} + 192 \sum m'_{ij} m'_{il} \Omega'_{jt} \Omega'_{lt} \kappa_{2i} \kappa_{2j} \kappa_{2l} \bar{\kappa}_{2t} \\ & + 288 \sum m'_{ij} \Omega'_{it} \Omega'_{ju} \Gamma'_{tu} \kappa_{2i} \kappa_{2j} \bar{\kappa}_{2t} \bar{\kappa}_{2u} + 192 \sum \Omega'_{it} \Omega'_{iu} \Gamma'_{tv} \Gamma'_{uv} \kappa_{2i} \bar{\kappa}_{2t} \bar{\kappa}_{2u} \bar{\kappa}_{2v} \\ & + 48 \sum \Gamma'_{tu} \Gamma'_{tv} \Gamma'_{uw} \Gamma'_{vw} \bar{\kappa}_{2t} \bar{\kappa}_{2u} \bar{\kappa}_{2v} \bar{\kappa}_{2w} + 192 \sum m'_{ij} m'_{il} \delta'_j \delta'_l \kappa_{2i} \kappa_{2j} \kappa_{2l} \\ & + 192 \sum \Omega'_{it} \Omega'_{jt} \delta'_i \delta'_j \kappa_{2i} \kappa_{2j} \bar{\kappa}_{2t} + 384 \sum m'_{ij} \Omega'_{it} \delta'_j \Lambda'_t \kappa_{2i} \kappa_{2j} \bar{\kappa}_{2t} \\ & + 192 \sum \Omega'_{it} \Omega'_{iu} \Lambda'_t \Lambda'_u \kappa_{2i} \bar{\kappa}_{2t} \bar{\kappa}_{2u} + 384 \sum \Omega'_{it} \Gamma'_{tu} \delta'_i \Lambda'_u \kappa_{2i} \bar{\kappa}_{2t} \bar{\kappa}_{2u} \\ & + 192 \sum \Gamma'_{tu} \Gamma'_{tv} \Lambda'_u \Lambda'_v \bar{\kappa}_{2t} \bar{\kappa}_{2u} \bar{\kappa}_{2v} .\end{aligned}$$

For ease of printing each summation sign stands for one, two, three or four separate summations as required by the subscripts. In these summations i, j, l and k run from 1 to n, t, u, v and w run from $(s - p + r + 1)$ to s. A further simplification will be to let δ'_i be zero and the summations for t, u, v and w run from $(s - p)$ to s. In this latter case the alternative hypotheses to that tested specify the existence of certain random variables but not any parameters.

7. Special cases of correlated variables. As an illustration of the use of the foregoing theory when the variables are correlated we consider the test for the linearity of regression in a bivariate table. The standard case where departure from linearity is represented by parameters was studied in [1]. It will now be supposed that the deviations from linearity form a simple moving average series of random variables. Let x_{ti} be the dependent variable and W_t the independent variable $(i = 1, \cdots, n_t ; t = 1, \cdots, s)$. We suppose that the model is

$$x_{ti} = \theta_1 + (W_t - \overline{W})\theta_2 + y_T + R y_{T-1} + z_{ti} ,$$

where $T = t + 2$ and R is a known constant. We shall assume that $\kappa_r(z_{ti}) = \kappa_{rt}$ (i.e., the distribution of z_{ti} depends only on the array). The fundamental sums of the squares are

$$S_a = \sum_t \sum_i (x_{ti} - \bar{x}_{t.})^2, \qquad S_r = \sum_t \sum_i \{x_{ti} - \bar{x}_{..} - b(W_t - \overline{W})\}^2,$$

where

$$b = \frac{\sum_t n_t(W_t - \overline{W})(\bar{x}_{t.} - \bar{x}_{..})}{\sum_t n_t(W_t - \overline{W})^2} .$$

Evaluation of the determinants gives

$$\Gamma'_{TT} = n_t + R^2 n_{t+1} - N^{-1}(n_t + Rn_{t+1})^2 - (\sum n_t w_t^2)^{-1}(n_t w_t + Rn_{t+1}w_{t+1})^2,$$

where $w_t = W_t - \overline{W}$ and $n_{s+1} = 0$. We have, therefore, using the determinants α_{ij} which have been worked out in [1],

$$\mathcal{E}(S_r) = \sum n_t \left(1 - \frac{1}{N} - \frac{w_t^2}{\sum n_t w_t^2}\right)\kappa_{2t}$$

$$+ \sum \left[n_t + R^2 n_{t+1} - \frac{(n_t + Rn_{t+1})^2}{N} - \frac{(n_t w_t + Rn_{t+1} w_{t+1})^2}{\sum n_t w_t^2}\right]\bar{\kappa}_{2T}$$

with the convention that $n_{s+1} = 0$. Again it may be shown that

$$\Gamma'_{T,T+1} = Rn_{t+1} - \frac{(n_t + Rn_{t+1})(n_{t+1} + Rn_{t+2})}{N}$$

$$- \frac{(n_t w_t + Rn_{t+1} w_{t+1})(n_{t+1} w_{t+1} + Rn_{t+2} w_{t+2})}{\sum n_t w_t^2}$$

and

$$\Gamma'_{TU} = -\frac{(n_t + Rn_{t+1})(n_u + Rn_{u+1})}{N} - \frac{(n_t w_t + Rn_{t+1} w_{t+1})(n_u w_u + Rn_{u+1} w_{u+1})}{\sum n_t w_t^2}$$

where $U = u + 2$ $(u = 1, \cdots s)$ and $|T - U| = |t - u| > 1$. Also if in terms of our original notation (Sections 2–6) i is in the tth group,

$$\Omega'_{iT} = 1 - \frac{n_t + Rn_{t+1}}{N} - \frac{w_t(n_t w_t + Rn_{t+1} w_{t+1})}{\sum n_t w_t^2},$$

if i is in the $(t + 1)$th group,

$$\Omega'_{iT} = R - \frac{n_t + Rn_{t+1}}{N} - \frac{w_t(n_t w_t + Rn_{t+1} w_{t+1})}{\sum n_t w_t^2},$$

and if i is not in the tth or the $(t + 1)$th groups,

$$\Omega'_{iT} = -\frac{n_t + Rn_{t+1}}{N} - \frac{w_t(n_t w_t + Rn_{t+1} w_{t+1})}{\sum n_t w_t^2}.$$

For brevity we write

$$\phi_{tu} = \frac{1}{N} + \frac{w_t w_u}{\sum n_t w_t^2},$$

$$\chi_{tu} = \frac{(n_t + Rn_{t+1})(n_u + Rn_{u+1})}{N} + \frac{(n_t w_t + Rn_{t+1} w_{t+1})(n_u w_u + Rn_{u+1} w_{u+1})}{\sum n_t w_t^2},$$

$$\psi_t = \frac{n_t + Rn_{t+1}}{N} + \frac{w_t(n_t w_t + Rn_{t+1} w_{t+1})}{\sum n_t w_t^2}.$$

Then

$$
\begin{aligned}
\kappa(S_r^2) = {} & \sum n_t(1 - \phi_{tt})^2\kappa_{4t} + \sum n_t(1 - 2\,\phi_{tt})\kappa_{2t}^2 + 2 \sum n_t n_u \phi_{tu}\kappa_{2t}\kappa_{2u} \\
& + \sum (n_t + R^2 n_{t+1} - \chi_{tt})^2 \bar{\kappa}_{4T} + 2 \sum (n_t + R^2 n_{t+1})(n_t + R^2 n_{t+1} - 2\chi_{tt})\bar{\kappa}_{2T}^2 \\
& + 4 \sum R n_{t+1}(R n_{t+1} - 2\chi_{t,t+1})\bar{\kappa}_{2T}\bar{\kappa}_{2,T+1} + 2 \sum \chi_{tu}^2 \kappa_{2T}\kappa_{2U} \\
& + \sum [n_t(1 - 2\psi_t)\kappa_{2t} + n_{t+1}R(1 - 2R\psi_t)\kappa_{2,t+1} + \sum n_u\psi_u\kappa_{2u}]\bar{\kappa}_{2T}.
\end{aligned}
$$

The higher cumulants follow in a similar way.

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