

# CONFIDENCE BOUNDS FOR A SET OF MEANS

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**1. Summary.** Professor John Tukey suggested the following two problems to the author: given that  $X_1, X_2, \dots, X_n$  are normally and independently distributed with unknown means  $\mu_1, \mu_2, \dots, \mu_n$  and given variance  $\sigma^2$ ;

PROBLEM A: Find a  $\beta$ -level confidence interval of the form

$$g(x_1, \dots, x_n) \geq \mu_1, \dots, \mu_n \geq -\infty.$$

PROBLEM B: Find a  $\beta$ -level confidence interval of the form

$$g(x_1, \dots, x_n) \geq \mu_1, \dots, \mu_n \geq h(x_1, \dots, x_n).$$

The main result of this paper is the nonexistence of intervals satisfying mild regularity conditions and having an exact confidence level (unless  $n = 1$  or  $\beta = 0, 1$ ). However for each problem an interval is given for which the confidence level is greater than or equal to  $\beta$  (formulas (2.1), (4.1)); these intervals are apparently shorter than those previously used in practice. Also the procedure for obtaining any interval with at least  $\beta$  confidence is described.

Some results are discussed for distributions other than the normal.

## 2. Introduction to Problem A.

**2.1. Normal distributions.** If  $X_1, \dots, X_n$  are normally and independently distributed with known variance  $\sigma^2$  and unknown means  $\mu_1, \dots, \mu_n$ , then Problem A is to find an upper  $\beta$ -level confidence bound for the set  $\{\mu_1, \dots, \mu_n\}$ ; that is, to find a function  $g(x_1, \dots, x_n)$  such that  $\Pr\{g(X_1, \dots, X_n) \geq \max \mu_i\} \geq \beta$  for all  $\mu_1, \dots, \mu_n$ .

One approach to this problem is to look for exact  $\beta$ -level confidence bounds: the above condition on the function  $g(x_1, \dots, x_n)$  is replaced by  $\Pr\{g(X_1, \dots, X_n) \geq \max \mu_i\} = \beta$  for all  $\mu_1, \dots, \mu_n$ . This more restrictive condition in a confidence region problem is of course analogous to the requirement of similarity in the theory of hypothesis testing.

In Section 3 Problem A is analyzed but attention is confined to measurable functions  $g(x_1, \dots, x_n)$  which satisfy two mild restrictions. These restrictions are given by the following assumptions concerning the function  $g(x_1, \dots, x_n)$ .

ASSUMPTION 2.1. For all  $x_1, \dots, x_n$ ,  $g(x_1 + \delta, \dots, x_n + \delta)$  is a monotone nondecreasing function of  $\delta$ .

ASSUMPTION 2.2. If  $x_j = \max x_i$  ( $i = 1, \dots, n$ ), then  $g(x_1, \dots, x_n)$  satisfies

$$g(x_1, \dots, x_n) \leq g(x_1, \dots, x_{j-1}, x_j + \delta, x_{j+1}, \dots, x_n)$$

for all  $x_1, \dots, x_n$  and for any positive  $\delta$ .

The second assumption seems reasonable since a bound would certainly be

suspect if it were smaller for 27.2, 25.5, 26.3, 27.8 than for 27.2, 25.5, 26.3, 27.5.

It is then proved by Theorem 1 that there does not exist an exact  $\beta$ -level confidence bound which satisfies these two assumptions.

As a by-product of Theorem 1 a bound having at least  $\beta$  confidence is obtained; it is

$$(2.1) \quad g(x_1, \dots, x_n) = \max x_i + N_{1-\beta} \sigma,$$

where  $N_{1-\beta}$  is the  $1 - \beta$  point of the unit normal; that is,

$$\frac{1}{\sqrt{2\pi}} \int_{N_\alpha}^{\infty} e^{-\frac{1}{2}x^2} dx = \alpha.$$

The optimum properties of this bound will be discussed in a later paper.

The above bound, however, is not the only confidence bound. In Section 3 a procedure is given for constructing bounds having at least  $\beta$  confidence. For this it is convenient to restrict attention to bounds satisfying a more restrictive version of Assumption 2.1. This assumption 2.1\* is obtained by applying the principle of cogredience to the problem using the transformations  $x'_i = x_i + C$ ,  $i = 1, \dots, n$ , for all  $C$ .

ASSUMPTION 2.1\*. *The function  $g(x_1, \dots, x_n)$  satisfies the equality*

$$g(x_1 + \delta, \dots, x_n + \delta) = g(x_1, \dots, x_n) + \delta$$

for all  $x_1, \dots, x_n, \delta$ .

2.2. *The general problem.* The problem as described above is a particular case of the following: given  $X_1, \dots, X_n$  are independently distributed with probability density functions  $f(x - \mu_1), \dots, f(x - \mu_n)$ , find a  $\beta$ -level confidence bound for the set  $\{\mu_1, \dots, \mu_n\}$ . Theorem 2 shows that if  $f(x - \mu)$  satisfies a condition of bounded completeness, then exact  $\beta$ -level bounds do not exist. A bound having at least  $\beta$  confidence can of course always be obtained by adding to  $\max x_i$  the  $1 - \beta$  point of the distribution  $f(x)$  (that is, with  $\mu = 0$ ).

### 3. Analysis of Problem A.

3.1. *Characteristic function of a confidence bound.* We define a characteristic function for the bound  $g(x_1, \dots, x_n)$  as follows:

$$(3.1) \quad \begin{aligned} \phi_\theta(x_1, \dots, x_n) &= 1, & g(x_1, \dots, x_n) &\geq \theta, \\ &= 0, & g(x_1, \dots, x_n) &< \theta. \end{aligned}$$

From assumption 2.1 we can infer that  $\phi_\theta(x_1 + \delta, \dots, x_n + \delta)$  is a monotone nondecreasing function of  $\delta$ .

To derive conditions on  $\phi_\theta(x_1, \dots, x_n)$  from Assumption 2.2 we first define disjoint sets  $S_1, \dots, S_n$  which cover  $R^n$  except for a set of measure zero:

$$(3.2) \quad S_i = \{(x_1, \dots, x_n) \mid x_i > \max_{j \neq i} x_j\}.$$

The second assumption insures that for points  $(x_1, \dots, x_n) \in S_i$   $\phi_\theta(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$  is a monotone nondecreasing function of  $x_i$ .

3.2. *Theorem for normal variables.* To prove Theorem 1 we shall need the following

LEMMA 1. *If  $Y_1, \dots, Y_n$  are normally and independently distributed with means  $\mu_1, \dots, \mu_n$  and unit variances, then the set of densities corresponding to all  $(\mu_1, \dots, \mu_n) \in [-\infty, 0]^n$  is boundedly complete; that is,*

$$E\{\phi(Y_1, \dots, Y_n)\} \equiv 0, \quad (\mu_1, \dots, \mu_n) \in [-\infty, 0]^n,$$

and

$$|\phi(y_1, \dots, y_n)| < M$$

imply

$$\phi(y_1, \dots, y_n) = 0$$

almost everywhere.

PROOF: The above set of distributions is complete; see Lehmann and Scheffé [1]. Since completeness implies bounded completeness, the lemma follows.

THEOREM 1. *If  $X_1, \dots, X_n$  are normal and independent with means  $\mu_1, \dots, \mu_n$  and variance  $\sigma^2$ , there does not exist (unless  $\beta = 0, 1$ , or  $n = 1$ ) a measurable function  $g(x_1, \dots, x_n)$ , satisfying assumptions 2.1 and 2.2, which is an exact  $\beta$  confidence bound for  $\max \mu_i$ .*

PROOF: Without loss of generality let  $\sigma^2 = 1$ . We consider a measurable function  $g(x_1, \dots, x_n)$  satisfying assumptions 2.1 and 2.2 and, assuming that  $g$  is an exact  $\beta$ -level confidence bound, we shall find that a contradiction results.

We have

$$\begin{aligned} \beta &= \Pr\{g(X_1, \dots, X_n) \geq \max \mu_i\} \\ (3.3) \quad &= E\{\phi_\theta(X_1, \dots, X_n) \mid \max \mu_i = \theta\} \\ &= E\{\beta_\theta^{(i)}(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n) \mid \max_{j \neq i} \mu_j < \theta\}, \end{aligned}$$

where

$$(3.4) \quad \beta_\theta^{(i)}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_\theta(x_1, \dots, x_n) e^{-\frac{1}{2}(x_i - \theta)^2} dx_i.$$

We now derive conditions on the function  $\beta_\theta^{(i)}$  and for simplicity let  $\theta = 0$ . From the expression above it is seen that

$$E\{\beta_0^{(i)}(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n) - \beta \mid \max_{j \neq i} \mu_j < 0\} = 0;$$

hence from Lemma 1, we conclude that

$$(3.5) \quad \beta_0^{(i)}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \beta$$

almost everywhere.

Using the above condition on  $\beta_0^{(i)}$ , we obtain conditions on the function  $\phi_0(x_1, \dots, x_n)$ .

$$(3.6) \quad \begin{aligned} \beta &= \beta_0^{(i)}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \quad \text{almost everywhere} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_0(x_1, \dots, x_n) e^{-\frac{1}{2}x_i^2} dx_i. \end{aligned}$$

Consider fixed  $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$  (not of course belonging to the exceptional set of measure zero for which the equality (3.5) might not hold). For  $x_i > \max_{j \neq i} x_j$ ,  $\phi_0(x_1, \dots, x_n)$  is a monotone function; and since it is a characteristic function it will have the following form

$$(3.7) \quad \begin{aligned} \phi_0(x_1, \dots, x_n) &= 0, & \max_{j \neq i} x_j < x_i < u(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \\ &= 1, & u(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) < x_i < \infty. \end{aligned}$$

$u(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  is taken to be the value of  $x_i$  at which  $\phi_0(x_1, \dots, x_n)$  jumps from 0 to 1 or  $\max_{j \neq i} x_j$ , whichever is larger. Using the function  $u(x_1, \dots, x_n)$ , we obtain

$$(3.8) \quad \begin{aligned} \beta &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\max_{j \neq i} x_j} \phi_0(x_1, \dots, x_n) e^{-\frac{1}{2}x_i^2} dx_i \\ &\quad + \frac{1}{\sqrt{2\pi}} \int_{u(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)}^{\infty} e^{-\frac{1}{2}x_i^2} dx_i. \end{aligned}$$

However, since

$$\begin{aligned} 0 &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\max_{j \neq i} x_j} \phi_0(x_1, \dots, x_n) e^{-\frac{1}{2}x_i^2} dx_i, \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\max_{j \neq i} x_j} e^{-\frac{1}{2}x_i^2} dx_i \\ &= \Pr\{X_i \leq \max_{j \neq i} x_j\}, \end{aligned}$$

then

$$(3.9) \quad N_\beta \leq u(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \leq N_{\beta - P(\max_{j \neq i} x_j)},$$

where

$$P(\max_{j \neq i} x_j) = \Pr\{X_i \leq \max_{j \neq i} x_j\}.$$

The inequality on  $u(x_1, \dots, x_n)$  implies that  $\phi_0(x_1, \dots, x_n)$  is equal to zero for almost all points in  $\mathcal{S}_i$  having  $x_i < N_\beta$ . This is true for all  $i$ ; hence  $\phi_0(x_1, \dots, x_n) = 0$  if  $\max x_j < N_\beta$ . Consider now  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  having  $\max_{j \neq i} x_j < N_\beta$ ; in expression (3.8), the first integral vanishes leaving

$$\beta = \frac{1}{\sqrt{2\pi}} \int_{u(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)}^{\infty} e^{-\frac{1}{2}x_i^2} dx_i.$$

Therefore

$$u(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = N_\beta, \quad \max_{j \neq i} x_j < N_\beta.$$

From the above equality on  $u(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ , we obtain the following conditions on  $\phi_0(x_1, \dots, x_n)$ :

$$\begin{aligned} \phi_0(x_1, \dots, x_n) &= 0, && \text{if } \max_1^n x_i < N_\beta, \\ &= 1, && \text{if exactly one } x \text{ is larger than } N_\beta. \end{aligned}$$

But since  $\phi_0(x_1 + \delta, \dots, x_n + \delta)$  is monotone in  $\delta$ , we have

$$(3.10) \quad \begin{aligned} \phi_0(x_1, \dots, x_n) &= 0, && \text{if } \max x_i < N_\beta, \\ &= 1, && \text{if } \max x_i > N_\beta. \end{aligned}$$

Therefore

$$\begin{aligned} g(x_1, \dots, x_n) &< 0, && \text{if } \max x_i < N_\beta, \\ &\geq 0, && \text{if } \max x_i > N_\beta. \end{aligned}$$

Similarly

$$(3.11) \quad \begin{aligned} g(x_1, \dots, x_n) &< \theta, && \text{if } \max x_i < N_\beta + \theta, \\ &\geq \theta, && \text{if } \max x_i > N_\beta + \theta. \end{aligned}$$

This completely determines  $g(x_1, \dots, x_n)$ ;

$$(3.12) \quad \begin{aligned} g(x_1, \dots, x_n) &= \max x_i - N_\beta \\ &= \max x_i + N_{1-\beta}. \end{aligned}$$

However, contrary to our original assumption, this function  $g(x_1, \dots, x_n)$  is not an exact  $\beta$ -level confidence bound unless  $\beta = 0, 1$ , or  $n = 1$ . For consider  $\beta_0^{(1)}(x_2, \dots, x_n)$ ; (3.5) gives

$$\beta_0^{(1)}(x_2, \dots, x_n) = \beta$$

almost everywhere, while the functional form of  $g(x_1, \dots, x_n)$  above implies

$$\begin{aligned} \beta_0^{(1)}(x_2, \dots, x_n) &= \beta, && \max_{j \neq 1} x_j \leq N_\beta, \\ &= 1, && \max_{j \neq 1} x_j > N_\beta. \end{aligned}$$

These are obviously in conflict unless  $n = 1$  or  $\beta = 0, 1$ . This completes the proof of Theorem 1.

3.3. *Examples of normal confidence bounds.* Although an exact  $\beta$ -level confidence bound satisfying assumptions 2.1 and 2.2 does not exist, bounds with at least  $\beta$  confidence do exist; an example of one was obtained in the course of the proof of Theorem 1, namely,

$$g(x_1, \dots, x_n) = \max x_i + N_{1-\beta} \sigma.$$

It is easily seen from the form of  $\beta_0^{(1)}(x_2, \dots, x_n)$  that this bound has at least  $\beta$  confidence,

$$\begin{aligned} \beta_0^{(1)}(x_2, \dots, x_n) &= \beta, & \max_{j \neq i} x_j &\leq N_\beta, \\ &= 1, & \max_{j \neq i} x_j &> N_\beta. \end{aligned}$$

The confidence level is

$$E\{\beta_0^{(1)}(X_2, \dots, X_n) \mid \max_{j \neq 1} \mu_j \leq 0\} \geq \beta.$$

We define a bound  $g$  having confidence at least  $\beta$  to be uniformly better than a bound  $g'$  having confidence at least  $\beta$ , if  $g \leq g'$  for all  $(x_1, \dots, x_n)$ , and  $g < g'$  on a set of positive measure. It is not difficult to see that bounds, uniformly better than the example above, do not exist. This obtains from the following simple property of the normal distribution. Let  $Y$  be normal with mean  $\theta$  and variance 1; then for  $\delta$  positive, all the probability less than  $C$  can be made arbitrarily small with respect to the probability in any small neighborhood of  $c + \delta$  by taking  $\theta$  large enough.

Since it may be desirable to obtain bounds other than the example given above, we outline the procedure. For spherically symmetric normal distributions in  $R^n$  having variance  $\sigma^2$  and mean  $(\mu_1, \dots, \mu_n)$  with  $\max \mu_i = 0$ , we look for a region whose size is greater than or equal to  $\beta$  and whose characteristic function  $\phi(x_1 + \delta, \dots, x_n + \delta)$  is monotone nondecreasing in  $\delta$ ; then a  $\beta$  level bound  $g(x_1, \dots, x_n)$  satisfying Assumption 2.1\* is the following:

$$g(x_1, \dots, x_n) = \delta'$$

where  $\delta'$  is the value of  $\delta$  at which  $\phi_0(x_1 - \delta, \dots, x_n - \delta)$  jumps from 0 to 1.

3.4. *Bounds for nonnormal distributions.* As remarked in Section 2, confidence bounds for  $\max \mu_i$  may be wanted for distributions other than the normal;  $\mu_1, \dots, \mu_n$  would of course be values of the location parameter corresponding to the random variables  $X_1, \dots, X_n$ . Consider the density function  $f(x - \mu)$ ; we shall say it is boundedly complete (one-sided) if

$$\int_{-\infty}^{\infty} g(x)f(x - \mu) dx = 0$$

for any dense set of  $\mu < 0$  and  $|g(x)| < M$  imply  $g(x) = 0$  almost everywhere. From a theorem of Lehmann and Scheffé which was mentioned in [1], we can conclude that if  $f(x - \mu)$  is boundedly complete (one-sided) then

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g(x_1, \dots, x_n) \prod_i f(x_i - \mu_i) \prod_i dx_i = 0$$

for all  $\mu_1, \dots, \mu_n < 0$  and  $|g(x_1, \dots, x_n)| < M$  imply  $g(x_1, \dots, x_n) = 0$  almost everywhere. This conclusion takes the place of Lemma 1 for the following theorem:

**THEOREM 2.** *If  $X_1, \dots, X_n$  are independent and have probability density functions  $f(x - \mu_1), \dots, f(x - \mu_n)$ , where  $f(x - \mu)$  is bounded complete (one-sided), then there does not exist (unless  $\beta = 0, 1$ , or  $n = 1$ ) a measurable function  $g(x_1, \dots, x_n)$ , satisfying Assumptions 2.1 and 2.2, which is an exact  $\beta$  confidence bound for  $\max \mu_i$ .*

**PROOF:** The proof is essentially that of Theorem 1.

**4. Introduction to Problem B.** The second problem is to find a confidence interval for a set of means; if  $X_1, \dots, X_n$  are normally and independently distributed with known variance  $\sigma^2$  and unknown means  $\mu_1, \dots, \mu_n$ , Problem B is to find two functions  $g(x_1, \dots, x_n), h(x_1, \dots, x_n)$  such that

$$\Pr\{g(X_1, \dots, X_n) \geq \mu_1, \dots, \mu_n \geq h(X_1, \dots, X_n)\} \geq \beta.$$

We also study the problem of finding an exact  $\beta$ -level confidence interval for which the above condition is replaced by

$$\Pr\{g(X_1, \dots, X_n) \geq \mu_1, \dots, \mu_n \geq h(X_1, \dots, X_n)\} = \beta.$$

In Section 5.3 we establish the nonexistence of exact  $\beta$ -level confidence intervals among pairs of functions  $(h, g)$  satisfying several moderate and reasonable restrictions; these restrictions are:

**ASSUMPTION 4.1.** *The functions  $g(x_1, \dots, x_n)$  and  $h(x_1, \dots, x_n)$  satisfy the equations*

$$g(x_r + \delta, \dots, x_n + \delta) = g(x_1, \dots, x_n) + \delta,$$

$$h(x_1 + \delta, \dots, x_n + \delta) = h(x_1, \dots, x_n) + \delta$$

for all  $x_1, \dots, x_n, \delta$ .

**ASSUMPTION 4.2.** *The equation*

$$g(x_1, \dots, x_n) = -h(-x_1, \dots, -x_n)$$

holds for all  $x_1, \dots, x_n$ .

**ASSUMPTION 4.3.** *The functions  $g(x_1, \dots, x_n)$  and  $h(x_1, \dots, x_n)$  are symmetric functions.*

**ASSUMPTION 4.4.** *If  $x_j = \max x_i$ , then the function  $g(x_1, \dots, x_n)$  satisfies*

$$g(x_1, \dots, x_n) \leq g(x_1, \dots, x_{j-1}, x_j + \delta, x_{j+1}, \dots, x_n)$$

for any positive  $\delta$ .

**ASSUMPTION 4.5.** *For all  $x_1, \dots, x_n, g(x_1, \dots, x_n) \geq \bar{x} + \epsilon_g$ , where  $\bar{x} = \sum x_i/n$  and  $\epsilon_g > 0$ , may depend on  $g$  but not on  $x_1, \dots, x_n$ .*

As a corollary to Theorem 3 we obtain a confidence interval for the means which has at least  $\beta$  confidence; it is (4.1)  $(h, g) = (\min x_i - N_{\frac{1}{2}(1-\beta)} \sigma, \max x_i + N_{\frac{1}{2}(1-\beta)} \sigma)$  where  $N_\alpha$  is the  $\alpha$  point of the unit normal. Also in section 5 we indicate the procedure for constructing intervals having at least  $\beta$  confidence.

**5. Analysis of Problem B.**

5.1. *Justification of assumptions.* The first three assumptions (4.1, 4.2 and 4.3) are obtained by applying the principle of cogredience to the problem. The set of transformations

$$x'_i = x_i + C, i = 1, \dots, n, \quad C \in R^1$$

produces the conditions contained in Assumption 4.1. Similarly the transformations

$$\begin{aligned} x'_i &= -x_i, i = 1, \dots, n, \\ x'_i &= x_{j_i}, i = 1, \dots, n, \end{aligned}$$

for all permutations  $(j_1, \dots, j_n)$  of  $(1, \dots, n)$  produce respectively the conditions of Assumptions 4.2 and 4.3.

Assumption 4.4 is similar in form and justification to Assumption 2.2. Assumption 4.5 is not too restrictive for practical confidence intervals: it is introduced merely from necessity in the proof. Nevertheless, it seems reasonable to suppose that Assumption 4.5 is not essential for the conclusions of the theorem.

5.2. *Characteristic functions.* A characteristic function similar to (3.1) could be defined for the interval  $(h, g)$ . However, the symmetry introduced by Assumption 4.2 enables us to use the characteristic function (3.1); for  $g(x_1, \dots, x_n)$  in  $(h, g)$  we define  $\phi_\theta(x_1, \dots, x_n)$  as in (3.1).

The present assumptions yield for  $\phi_\theta(x_1, \dots, x_n)$  the properties derived in Section 3.1, namely,

- (1)  $\phi_\theta(x_1 + \delta, \dots, x_n + \delta)$  is monotone nondecreasing as a function of  $\delta$ , and
- (2) for points  $(x_1, \dots, x_n) \in S_i$ ,  $\phi_\theta(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$  is a monotone nondecreasing function of  $x_i$ .

5.3. *Theorem for normal distributions.* To establish the nonexistence of exact  $\beta$ -level confidence intervals satisfying Assumptions 4.1 to 4.5, we have

**THEOREM 3.** *If  $X_1, \dots, X_n$  are normally and independently distributed with means  $\mu_1, \dots, \mu_n$  and variance  $\sigma^2$ , there does not exist (unless  $\beta = 0, 1$  or  $n = 1$ ) a pair of measurable functions  $(g, h)$  which satisfies Assumptions 4.1, 4.2, 4.3, 4.4, 4.5 and which is an exact  $\beta$ -level confidence interval for the set  $\{\mu_1, \dots, \mu_n\}$ .*

**PROOF:** The proof is somewhat different from that used in Theorem 1, but several results obtained in the course of that proof are used here.

Let  $\sigma^2 = 1$  without loss of generality. We consider a pair of functions  $(h, g)$  satisfying Assumptions 4.1 to 4.5, and, assuming  $(h, g)$  is an exact  $\beta$  level confidence interval, we shall find that a contradiction results.

For the characteristic function  $\phi_\theta(x_1, \dots, x_n)$  define according to (3.4) a conditional expectation  $\beta_\theta^{(i)}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ . In the following expressions we shall use a symmetric multivariate normal distribution with variance 1 and mean given after the condition bars. Using Assumption 4.1, we have

$$\begin{aligned} \beta &= \Pr\{g(X_1, \dots, X_n) \geq 0, h(X_1 + \theta_{n-1}, \dots, X_n + \theta_{n-1}) \\ &\leq 0 \mid (\theta, -\theta_1, \dots, -\theta_{n-1})\} \end{aligned}$$



if we let  $0 \leq \theta_1 \leq \dots \leq \theta_{n-1}$ . Using Assumptions 4.1, 4.2, 4.3,

$$\begin{aligned} 1 - \beta &= E\{(1 - \phi_0(X_1, \dots, X_n)) \mid (0, -\theta_1, \dots, -\theta_{n-1})\} \\ &\quad + E\{(1 - \phi_0(X_1 - \theta_{n-1}, \dots, X_n - \theta_{n-1})) \mid (0, \theta_1, \dots, \theta_{n-1})\} \\ &\quad - E\{(1 - \phi_0(X_1, \dots, X_n)) (1 - \phi_0(-X_1 - \theta_{n-1}, \dots, -X_n - \theta_{n-1})) \\ &\quad \qquad \qquad \qquad \mid (0, -\theta_1, \dots, -\theta_{n-1})\} \\ &= E\{(1 - \phi_0(X_1, \dots, X_n) \mid (0, -\theta_1, \dots, -\theta_{n-1})\} \\ &\quad + E\{(1 - \phi_0(X_1, \dots, X_n) \\ &\quad \qquad \qquad \mid (-\theta_{n-1}, -(\theta_{n-1} - \theta_1), \dots, -(\theta_{n-1} - \theta_{n-2}), 0)\} \\ &\quad - E\{(1 - \phi_0(X_1, \dots, X_n)) (1 - \phi_0(-X_1 - \theta_{n-1}, \dots, -X_n - \theta_{n-1})) \\ &\quad \qquad \qquad \mid (0, -\theta_1, \dots, -\theta_{n-1})\}. \end{aligned}$$

If we restrict the values of the  $\theta$ 's it is possible to make the third term on the right hand side of the equation equal to zero. From Assumption 4.5 the first factor  $1 - \phi_0(x_1, \dots, x_n)$  is equal to zero if  $g(x_1, \dots, x_n) \geq 0$  or if  $\bar{x} + \epsilon_\theta \geq 0$ . Similarly the second factor  $1 - \phi_0(-x_1 - \theta_{n-1}, \dots, -x_n - \theta_{n-1})$  is equal to zero, if  $-x_i - \theta_{n-1} + \epsilon_\theta \geq 0$  or if  $\bar{x} + \epsilon_\theta \leq 2\epsilon_\theta - \theta_{n-1}$ . The product of the two factor will certainly be zero if  $\theta_{n-1} < 2\epsilon_\theta$ . Therefore we have

$$\begin{aligned} 1 - \beta &= E\{(1 - \phi_0(x_1, \dots, x_n)) \mid (0, -\theta_1, \dots, -\theta_{n-1})\} \\ &\quad + E\{(1 - \phi_0(x_1, \dots, x_n)) \mid (0, -(\theta_{n-1} - \theta_{n-2}), \dots, -(\theta_{n-1} - \theta_1))\} \end{aligned}$$

for all  $\theta_1, \dots, \theta_{n-1}$  satisfying  $0 \leq \theta_1 \leq \dots \leq \theta_{n-1} < 2\epsilon_\theta$ .

We now derive a property of the conditional expectation  $\beta_0^{(1)}(x_2, \dots, x_n) = \beta(x_2, \dots, x_n)$ . We have

$$\begin{aligned} 1 - \beta &= \frac{1}{(2\pi)^{\frac{1}{2}(n-1)}} \int (1 - \beta(x_2, \dots, x_n)) \exp \left[ -\frac{1}{2} \left\{ \sum_2^n (x_i + \theta_{i-1})^2 \right\} \right] dx_2 \dots dx_n \\ &\quad + \frac{1}{(2\pi)^{\frac{1}{2}(n-1)}} \int (1 - \beta(x_2, \dots, x_n)) \\ &\quad \cdot \exp \left[ -\frac{1}{2} \left\{ \sum_2^{n-1} (x_i + \theta_{n-1} - \theta_{i-1})^2 + (x_n + \theta_{n-1})^2 \right\} \right] dx_2 \dots dx_n. \end{aligned}$$

We note that the following functions satisfy the conditions for a pdf in  $R^{n-1}$ :

$$\begin{aligned} f_1 &= \frac{1}{2(2\pi)^{\frac{1}{2}(n-1)}} \left\{ \exp \left[ -\frac{1}{2} \sum_2^n (x_i + \theta_{i-1})^2 \right] \right. \\ &\quad \left. + \exp \int \left[ -\frac{1}{2} \sum_2^{n-1} (x_i + \theta_{n-1} - \theta_{i-1})^2 - \frac{1}{2} (x_n + \theta_n - 1)^2 \right] \right\}, \end{aligned}$$

$$f_2 = 2 \frac{1 - \beta(x_2, \dots, x_n)}{1 - \beta} f_1.$$

For the integral

$$\int_{R^{n-1}} f_i dx_2 \cdots dx_n = 1,$$

the conditions are satisfied for differentiating any number of times under the sign of integration with respect to  $\theta_1, \dots, \theta_{n-1}$ . If we set  $\theta_1 = \dots = \theta_{n-1} = 0$  in the equation

$$\int \frac{\partial^{r_2 + \dots + r_n}}{\partial \theta_1^{r_2} \cdots \partial \theta_{n-1}^{r_n}} f_i dx_2 \cdots dx_n = 0,$$

we obtain equations from which all the moments of  $f_i$  (with all  $\theta$ 's equal zero) can be obtained. However, the equations do not depend on  $i$ ; hence  $f_1$  and  $f_2$  have identical moments. But for these multivariate normal moments, the density corresponding to them is unique (generalization of moment condition on p. 176 in [2]); therefore,

$$1 - \beta(x_2, \dots, x_n) = \frac{1}{2}(1 - \beta)$$

or

$$\beta(x_2, \dots, x_n) = \frac{1}{2}(1 + \beta) = \beta^*.$$

Now if we use part of the proof of Theorem 1 from formula (3.5) to formula (3.12), we obtain

$$\begin{aligned} g(x_1, \dots, x_n) &= \max x_i + N_{1-\beta^*}, \\ &= \max x_i + N_{\frac{1}{2}(1-\beta)}, \end{aligned}$$

and by Assumption 4.2, we have

$$h(x_1, \dots, x_n) = \min x_i - N_{\frac{1}{2}(1-\beta)}.$$

Therefore

$$(h, g) = (\min x_i - N_{\frac{1}{2}(1-\beta)}, \max x_i + N_{\frac{1}{2}(1-\beta)}).$$

It is easily seen that for this interval the confidence level is greater than  $\beta$  (unless  $\beta = 0, 1$ , or  $n = 1$ ). Since this is a contradiction the theorem is proved.

5.4. *Example of normal confidence intervals.* Intervals having at least  $\beta$  confidence do exist; for example

$$(\min x_i - \sigma N_{\frac{1}{2}(1-\beta)}, \max x_i + \sigma N_{\frac{1}{2}(1-\beta)}).$$

The confidence level for this interval is always larger than  $\beta$  and it seems reasonable to expect that it is bounded away from  $\beta$ . In other words the above interval might be refined by using a constant smaller than  $N_{\frac{1}{2}(1-\beta)}$ . The answer to this question will most likely be obtained only by applying a numerical procedure analogous to that described at the end of Section 3.4.

Any bounds for Problem A can be used to provide an interval for the present problem. Let  $1 - \beta = \alpha_1 + \alpha_2$  where  $\alpha_1$  and  $\alpha_2$  are positive, and let  $g_1(x_1, \dots, x_n)$ ,  $G_2(x_1, \dots, x_n)$  be at least  $1 - \alpha_1$ ,  $1 - \alpha_2$  confidence bounds for problem A. Then an interval having at least  $\beta$  confidence is

$$(-g_2(-x_1, \dots, -x_n), g_1(x_1, \dots, x_n))$$

This follows from the argument at the beginning of the proof of Theorem 3.

#### REFERENCES

- [1] E. LEHMANN AND H. SCHEFFÉ, "Completeness, similar regions, and unbiased estimation," *Sankhyā*, Vol. 10 (1950), pp. 305-340.
- [2] H. CRAMÉR, *Mathematical Methods of Statistics*, Princeton University Press, 1946.