ORTHOGONAL ARRAYS OF STRENGTH TWO AND THREE

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1. Summary. Orthogonal arrays can be regarded as natural generalizations of orthogonal Latin squares, and are useful in various problems of experimental design. In this paper the known upper bounds for the maximum possible number of constraints for arrays of strength 2 and 3 have been improved, and certain methods for constructing these arrays have been given.

2. Introduction. A $k \times N$ matrix A, with entries from a set Σ of $s \geqq 2$ elements, is called an orthogonal array of strength t, size N, k constraints and s levels if each $t \times N$ submatrix of A contains all possible $t \times 1$ column vectors with the same frequency λ. The array may be denoted by (N, k, s, t). The number λ may be called the index of the array. Clearly $N = \lambda s^t$.

The set Σ will for convenience be taken as the set of integers $0, 1, 2, \cdots, s - 1$. For example the orthogonal array (18, 7, 3, 2) with index 2 is given below. It is easy to verify that in any 2×18 submatrix, each of the column vectors (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2) occurs twice.

$$
\begin{array}{cccccccccccccccccc}
0&1&2&0&1&2&0&1&2&0&1&2&0&1&2&0&1&2\\
0&1&2&0&1&2&1&2&0&2&0&1&1&2&0&2&0&1\\
0&1&2&1&2&0&0&1&2&2&0&1&2&0&1&1&2&0\\
0&1&2&2&0&1&2&0&1&0&1&2&1&2&0&1&2&0\\
0&1&2&1&2&0&2&0&1&1&2&0&0&1&2&2&0&1\\
0&1&2&2&0&1&1&2&0&1&2&0&2&0&1&0&1&2\\
0&0&0&0&0&0&1&1&1&1&1&1&2&2&2&2&2&2
\end{array}
\tag{2.0}
$$

If the orthogonal array A is of strength t so is any subarray of k' rows (constraints) if $k' \leqq k$. Hence the non-existence of $(\lambda s^t, k', s, t)$ automatically implies the non-existence of $(\lambda s^t, k, s, t)$ if $k > k'$. Again if A is of strength t, it is also of strength t' for all $t' \leqq t$.

The optimum multifactorial designs considered by Plackett and Burman [1] are essentially orthogonal arrays of strength 2. They have shown that the maximum number of constraints k for an orthogonal array of size λs^2, s levels and strength 2, satisfies the inequality

$$k \leqq \left[\frac{\lambda s^2 - 1}{s - 1}\right], \tag{2.1}$$

where $[x]$ is largest possible integer not exceeding x. The square bracket is used in this sense throughout this paper.

The existence of the orthogonal array $(s^2, k, s, 2)$ is combinatorially equivalent to the existence of a set of $k - 2$ mutually orthogonal $s \times s$ Latin squares (such a set is usually said to have k constraints, represented by rows, columns and the $k - 2$ squares). The inequality (2.1) for the case $\lambda = 1$ states the well known fact that the maximum number of mutually orthogonal $s \times s$ Latin squares cannot exceed $s - 1$.

Again for the special case $s = 2$, (2.1) gives $k \leqq 4\lambda - 1$. Plackett and Burman have actually constructed orthogonal arrays $(4\lambda, 4\lambda - 1, 2, 2)$ for all values of $\lambda \leqq 25$, except for $\lambda = 23$. They also give a number of arrays of strength 2 for other values of s, and establish a connection between orthogonal arrays and affine resolvable balanced incomplete block designs [2], and between orthogonal arrays and partially balanced designs [3].

Rao [4] studies hypercubes of strength d, which are orthogonal arrays for which the index λ is a power of s. He has used them in connection with confounded factorial designs. The concept of orthogonal arrays in its most general form is also due to Rao [5]. He discusses the use of these arrays together with some methods of constructing them and gives the following generalization of the inequality of Plackett and Burman.

THEOREM. *For an orthogonal array* $(\lambda s^t, k, s, t)$, $t \geqq 2$, *the number of constraints* k *satisfies the inequality*

$$\lambda s^t - 1 \geqq C_1^k(s-1) + \cdots + C_u^k(s-1)^u \qquad \text{if } t = 2u, \tag{2.2}$$

$$\lambda s^t - 1 \geqq C_1^k(s-1) + \cdots + C_u^k(s-1)^u + C_u^{k-1}(s-1)^{u+1} \qquad \text{if } t = 2u+1. \tag{2.3}$$

When $t = 2$, this leads to Plackett and Burman's inequality (2.1). When $t = 3$, we get

COROLLARY. *For an orthogonal array* $(\lambda s^3, k, s, 3)$ *of strength* 3, *the number of constraints* k *satisfies the inequality*

$$k \leqq \left[\frac{\lambda s^2 - 1}{s - 1}\right] + 1. \tag{2.4}$$

Theorems 1A and 2A proved in Sections 3 and 4 give an alternative proof of the inequalities (2.1) and (2.4). Theorems 1B, 2B, 2C improve these inequalities except for certain special values of s.

Sections 5, 6 and 7 are devoted to the investigation of methods for constructing orthogonal arrays of strength 2. A difference theorem is proved, which when used in conjunction with Galois fields enables the construction of the arrays (18, 7, 3, 2) and (32, 9, 4, 2). The first of these has been constructed by Burman [1] by trial and error methods. It is shown that if p is prime and $s = p^v$, $\lambda = p^u$, $[u/v] = c$, then we can construct an orthogonal array $(\lambda s^2, k, s, 2)$, where $k = \{\lambda(s^{c+1} - 1)/(s^c - s^{c-1})\} + 1$.

Theorems 5A and 5B of Section 8 establish a connection between orthogonal arrays and the theory of confounding in symmetrical factorial designs (based on the use of finite projective geometries) first developed by Bose and Kishen [6] and later amplified by Bose [7]. It is shown that the problem of constructing the orthogonal array (s^r, k, s, t), $r \geqq t$, $s = p^n$ and the problem of obtaining a symmetrical factorial design with s levels and k factors, in which the block size is s^r and in which all t-factor and lower order interactions are left unconfounded, both depend on finding a set of k points in $PG(r - 1, p^n)$ no t of which are conjoint. Such sets have been obtained by Bose in [7] and his results can be immediately translated into the language of orthogonal arrays. This has been done in Section 9. Theorems 5A and 5B were given by Bush in his unpublished thesis [8]. It has recently come to our notice that Rao [9] independently obtained a theorem equivalent to 5A, and derived from it the array $(2^r, 2^{r-1}, 2, 3)$ given by us in Section 9(a). The results in paragraphs (b) and (c) of Section 9 are new. An improvement of the inequalities (2.2) and (2.3) has been given by Bush [8, 10] for the special case $\lambda = 1$.

3. Upper bound for the number of constraints for orthogonal arrays of strength 2. Two columns of an orthogonal array are said to have i coincidences if there are exactly i rows in which the symbols appearing in the two columns have the same value (i.e., are the same elements of Σ). For example, the first column in the array (2.0) has 1 coincidence with each of the second and third columns, but has 3 coincidences with the fourth.

For any orthogonal array (N, k, s, t) of index λ let n_i denote the number of columns (other than the first) which have i coincidences with the first column. Since the total number of columns is $N = \lambda s^t$,

$$\sum_{i=0}^{k} n_i = \lambda s^t - 1. \tag{3.0a}$$

We shall show that

$$\sum_{i=0}^{k} i(i-1)\cdots(i-h+1)n_i = k(k-1)\cdots(k-h+1)(\lambda s^{t-h} - 1), \quad 1 \leqq h \leqq t. \tag{3.0b}$$

The formula (3.0a) can be regarded as a degenerate case of (3.0b) for $h = 0$.

Consider the subarray obtained by choosing any h rows of (N, k, s, t). The first column vector of this array appears in exactly $\lambda s^{t-h} - 1$ other columns of this subarray. Since it is possible to choose the subarray in C_h^k different ways the total number of $h \times 1$ vectors appearing in columns other than the first which are identical with the corresponding vector of the first column is $(\lambda s^{t-h} - 1)C_h^k$. But any column which has i-coincidences with the first contributes nothing or C_h^i to this number according as $i < h$ or $i \geqq h$. Hence

$$\sum_{i=0}^{k} n_i C_h^i = C_h^k(\lambda s^{t-h} - 1), \tag{3.0c}$$

where C_h^i is to be interpreted as zero if $i < h$. This is equivalent to (3.0b).

Let us now confine our attention to orthogonal arrays of strength 2. Then (3.0a) and (3.0b) lead to

$$\sum_{i=0}^{k} n_i = \lambda s^2 - 1, \tag{3.1a}$$

$$\sum_{i=0}^{k} i n_i = k(\lambda s - 1), \tag{3.1b}$$

$$\sum_{i=0}^{k} i(i-1) n_i = k(k-1)(\lambda - 1). \tag{3.1c}$$

Consider the function

$$f(x) = \sum_{i=0}^{k} (i - x)(i - 1 - x) n_i,$$

defined for integral values of x. Then

$$f(x) \geqq 0 \tag{3.2}$$

since $n_i \geqq 0$, and the factors $(i - x)$ and $(i - 1 - x)$ are both negative if $i < x$, and both positive if $i > x + 1$. Also one factor is zero if $i = x$ or $x + 1$. Now

$$f(x) = \sum_{i=0}^{k} i(i-1) n_i - 2x \sum_{i=1}^{k} i n_i + x(x+1) \sum_{i=1}^{k} n_i;$$

whence from (3.1), we get

$$f(x) = \lambda\{k(k-1) - 2kxs + x(x+1)s^2\} - \{k(k-1) - 2kx + x(x+1)\}.$$

From (3.2)

$$\lambda \geqq \frac{k(k-1) - 2kx + x(x+1)}{k(k-1) - 2kxs + x(x+1)s^2}. \tag{3.3}$$

Setting

$$\alpha = k - 1 - xs,$$

we can after some reduction, write (3.3) in the form

$$\frac{\lambda s^2 - 1}{s - 1} \geqq k\left\{1 + \frac{\alpha(s - \alpha)}{D}\right\}, \tag{3.4}$$

where D can be expressed in two equivalent forms

$$D = (s-1)(k - \alpha - 1) + \alpha(\alpha + 1) \tag{3.5a}$$

$$= k(s-1) - (\alpha + 1)(s - 1 - \alpha). \tag{3.5b}$$

We shall now prove Plackett and Burman's inequality (2.1) for orthogonal arrays of strength 2, and then proceed to improve it if $\lambda - 1$ is not divisible by $s - 1$. Let

$$\lambda - 1 = a(s - 1) + b, \qquad 0 \leqq b < s - 1, \qquad a \geqq 0.$$

Therefore

$$\frac{\lambda s^2 - 1}{s - 1} = \lambda s + \lambda + a + \frac{b}{s - 1}. \tag{3.6}$$

Suppose there exists an array with $k = \lambda s + \lambda + a + 1$. Then

$$k - 1 = s(\lambda + a) + b + 1.$$

The integer x is at our disposal. Let us choose $x = \lambda + a$; then $\alpha = b + 1$, so that $0 < \alpha < s$.

From (3.5a) we have

$$D = s(s - 1)(\lambda + a) + \alpha(\alpha + 1) > 0;$$

so that

$$\frac{\alpha(s - \alpha)}{D} > 0.$$

Hence from (3.4) and (3.6)

$$\frac{b}{s - 1} > 1,$$

which is a contradiction. Hence, the value $k = \lambda s + \lambda + a + 1$ is inadmissible, and so are all higher values. This proves the inequality of Plackett and Burman.

THEOREM 1A. *For any orthogonal array* $(\lambda s^2, k, s, 2)$ *of strength* 2, *the number of constraints* k *satisfies the inequality*

$$k \leqq \left[\frac{\lambda s^2 - 1}{s - 1}\right].$$

Consider now the case when $\lambda - 1$ is not divisible by $s - 1$, so that $0 < b < s - 1$. Let

$$k = \lambda s + \lambda + a - n, \qquad b > n \geqq 0.$$

Therefore

$$k - 1 = s(\lambda + a) + b - n.$$

Choosing x as before, we now have

$$0 < \alpha = b - n < s - 1.$$

Therefore

$$(\alpha + 1)(s - 1 - \alpha) > 0.$$

Hence from (3.4) and (3.5b)

$$\frac{\lambda s^2 - 1}{s - 1} > k + \frac{\alpha(s - \alpha)}{(s - 1)},$$

or

$$\frac{b}{s-1} > -n + \frac{(b-n)(s-b+n)}{s-1}.$$

Therefore

$$(b-n)(b+1-n) - s(b-2n) > 0. \tag{3.7}$$

Hence if n is any integer ($b > n \geqq 0$) for which the relation (3.7) is contradicted then the value $k = \lambda s + \lambda + a - n$ and all higher values are impossible. The first term in (3.7) is never negative, so that for $n > b/2$, this relation will never be contradicted. Hence we may drop the restriction $b > n$. The quadratic equation obtained by replacing the inequality by equality in (3.7) has one positive and one negative root, since the product of the roots is $-b(s-1-b)$ and $0 < b < s-1$. The positive root may be written as

$$\theta = \frac{\sqrt{1 + 4s(s-1-b)} - (2s-2b-1)}{2}. \tag{3.8}$$

The largest value of n which contradicts (3.6) is $[\theta]$. Hence we may state the following theorem.

THEOREM 1B. *If* $\lambda - 1 = a(s-1) + b, 0 < b < s-1$, *then for the orthogonal array* $(\lambda s^2, k, s, 2)$ *of strength* 2, *the number of constraints* k *satisfies the inequality*

$$k \leqq \left[\frac{\lambda s^2 - 1}{\lambda - 1}\right] - [\theta] - 1, \tag{3.9}$$

where θ *is the positive number given by* (3.8).

4. Upper bound for the number of constraints for orthogonal arrays of strength 3. Consider an array of strength 3, and let n_i denote the number of columns (other than the first) which have i coincidences with the first column. From (3.0a) and (3.0b)

$$\sum_{i=0}^{k} n_i = \lambda s^3 - 1, \tag{4.0a}$$

$$\sum_{i=0}^{k} i n_i = k(\lambda s^2 - 1), \tag{4.0b}$$

$$\sum_{i=0}^{k} i(i-1) n_i = k(k-1)(\lambda s - 1), \tag{4.0c}$$

$$\sum_{i=0}^{k} i(i-1)(i-2) n_i = k(k-1)(k-2)(\lambda - 1). \tag{4.0d}$$

If x is any positive integer then

$$f(x) = \sum_{i=0}^{k} i(i-1-x)(i-2-x) n_i \geqq 0; \tag{4.1}$$

whence from (4.0) we get

$$f(x) = k(k-1)(k-2)(\lambda-1) - 2xk(k-1)(\lambda s-1) + kx(x+1)(\lambda s^2-1) \geqq 0. \tag{4.2}$$

Since $k \geqq 1$, we have

$$\lambda \geqq \frac{(k-1)(k-2) - 2x(k-1) + x(x+1)}{(k-1)(k-2) - 2x(k-1)s + x(x+1)s^2}, \tag{4.3}$$

which is the same as (3.3) with $k - 1$ instead of k. Hence reasoning as before we can prove the following theorems.

THEOREM 2A. *For any orthogonal array* $(\lambda s^3, k, s, 3)$ *of strength* 3, *the number of constraints* k *satisfies the inequality*

$$k \leqq \left[\frac{\lambda s^2 - 1}{s - 1}\right] + 1. \tag{4.4}$$

THEOREM 2B. *If* $\lambda - 1 = a(s-1) + b, 0 < b < s - 1$, *then for the orthogonal array* $(\lambda s^3, k, s, 3)$ *of strength* 3, *the number of constraints* k *satisfies the inequality*

$$k \leqq \left[\frac{\lambda s^2 - 1}{\lambda - 1}\right] - [\theta], \tag{4.5}$$

where θ *is the positive number given by* (3.8).

Theorem 2A is the same as the Rao inequality (2.4) and Theorem 2B improves it for the case when $\lambda - 1$ is not divisible by $s - 1$.

We shall now show that when $\lambda - 1$ is divisible by $s - 1$, we can still improve the inequality of Theorem 2A, except in certain special cases. In fact we can state the following theorem.

THEOREM 2C. *For any orthogonal array* $(\lambda s^3, k, s, 3)$ *of strength* 3, *if* $\lambda - 1 = a(s-1)$, *and* $(s-1)^2(s-2)$ *is not divisible by* $as + 2$ *then the number of constraints* k *satisfies the inequality*

$$k \leqq \left[\frac{\lambda s^2 - 1}{s - 1}\right] - 1. \tag{4.6}$$

Now $[(\lambda s^2 - 1)/(s-1)] = as^2 + s + 1$. If possible let $k = as^2 + s + 1$. Choose $x = as$. Then it is easy to verify from (4.2) that $f(x) = 0$. Hence

$$\sum_{i=0}^{k} i(i-1-as)(i-2-as)n_i = 0.$$

Since $n_i \geqq 0$, it follows that n_i must vanish for all values of i except $i = 0, as + 1, as + 2$. From (4.0b) and (4.0c) we get

$$(as+1)n_{as+1} + (as+2)n_{as+2} = k(as^3 - as^2 + s^2 - 1),$$

$$asn_{as+1} + (as+2)n_{as+2} = ks(as^2 - as + s - 1).$$

Solving we get

$$n_{as+1} = k(s-1),$$

$$n_{as+2} = \frac{ks(as^2 - 2as + a + s - 1)}{as + 2}$$

$$= k(s-1)^2 - s(s-1)(s-2) + \frac{(s-1)^2(s-2)}{as+2}.$$

Since n_{as+2} must be integral, we arrive at a contradiction if $(s-1)^2(s-2)$ is not divisible by $as + 2$. Hence in this case $k \leqq as^2 + s$.

Consider the special case $\lambda = s$. Then $a = 1$. If $(s-1)^2(s-2)/(s+2)$ is integral, then 36 must be divisible by $s + 2$. We can therefore state the following corollary to Theorem 2C.

COROLLARY. *For the orthogonal array* $(s^4, k, s, 3)$ *if* 36 *is not divisible by* $s + 2$, *then the number of constraints* k *cannot exceed* $s^2 + s$.

5. The method of differences for constructing orthogonal arrays of strength 2. The method of differences has been elsewhere used [11] for constructing incomplete block designs. Here we shall use it to construct orthogonal arrays of strength 2.

Let $\lambda = \alpha\beta$. An orthogonal array $(\lambda s^2, k, s, 2)$ of strength 2 is said to be β-resolvable if it is the juxtaposition of $g = \alpha s$ different arrays $(\beta s, k, s, 1)$ of index β and strength 1. A 1-resolvable array is said to be completely resolvable. For example, the array (18, 6, 3, 2), obtained from (2.0) by deleting the last row is completely resolvable.

If $\lambda = \alpha\beta$ and the orthogonal array $(\lambda s^2, k, s, 2)$ is β-resolvable, then we can add at least one more row and get an orthogonal array of $k + 1$ constraints. In the new row we have to put the first element of Σ in the columns belonging to the first component array, the second element of Σ in the columns belonging to the next component and so on. As will be seen later under appropriate circumstances, it may be possible to add more than one row without destroying the orthogonality of the array.

THEOREM 3. *Let* M *be a module (additive group) consisting of* s *elements*, e_0, $e_1, \cdots, e_{s-1}$. *Suppose it is possible to find a scheme of* r *rows, with elements belonging to* M

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rn} \end{matrix} \tag{5.0}$$

such that among the differences of the corresponding elements of any two rows, each element of M *occurs exactly* λ *times* $(n = \lambda s)$; *then the method of constructing a completely resolvable orthogonal array* $(\lambda s^2, r, s, 2)$ *of strength* 2 *is as follows: Write* *down the addition table of M. Then replace each element in the scheme by the row of the addition table corresponding to the element (using only the suffixes if the set* Σ *is taken as* $0, 1, \cdots, s-1$*). This gives the completely resolvable array* $(\lambda s^2, r, s, 2)$*. A new row can be added to obtain an array* $(\lambda s^2, r+1, s, 2)$ *of* $r+1$ *constraints.*

Before proceeding to a formal proof we shall illustrate the use of the theorem, by constructing the orthogonal array (18, 7, 3, 2). For M we take the Galois field GF (3), whose elements are residue classes (mod 3). Let $e_0 = 0$, $e_1 = 1$, $e_2 = 2$. The addition table of M is

$$\begin{array}{c|ccc} & e_0 & e_1 & e_2 \\ \hline e_0 & e_0 & e_1 & e_2 \\ e_1 & e_1 & e_2 & e_0 \\ e_2 & e_2 & e_0 & e_1 \end{array} \tag{5.1}$$

It is not difficult to construct by trial a six rowed scheme

$$\begin{array}{cccccc} e_0 & e_0 & e_0 & e_0 & e_0 & e_0 \\ e_0 & e_0 & e_1 & e_2 & e_1 & e_2 \\ e_0 & e_1 & e_0 & e_2 & e_2 & e_1 \\ e_0 & e_2 & e_2 & e_0 & e_1 & e_1 \\ e_0 & e_1 & e_2 & e_1 & e_0 & e_2 \\ e_0 & e_2 & e_1 & e_1 & e_2 & e_0 \end{array} \tag{5.2}$$

where among the differences of the corresponding elements in any two rows each of the three elements e_0, e_1, e_2 occurs twice. In order to convert the scheme (5.2) into the completely resolvable orthogonal array (18, 6, 3, 2), we replace each element of M by the suffixes in the corresponding row of the addition table (5.1). Thus

$$e_0 \rightarrow 0,\ 1,\ 2,$$
$$e_1 \rightarrow 1,\ 2,\ 0,$$
$$e_2 \rightarrow 2,\ 0,\ 1.$$

We thus obtain the first six rows of the array (2.0) given in the Introduction.

Finally to obtain the array (18, 7, 3, 2) we add a new row consisting of six zeros (occupying the columns of the first two groups) followed by six ones, followed by six twos. It should be noted that from Theorem 1B, 7 is the maximum possible number of constraints for an array of size 18 and strength 2, with 3 levels.

We shall now proceed to a formal proof of Theorem 3. The s^2 2×1 vectors whose components are elements of M can be divided into s classes, each class corresponding to one element of M. If $e_i - e_j = e_k$ then $\begin{pmatrix} e_i \\ e_j \end{pmatrix}$ belongs to the class corresponding to e_k. Now in the addition table of M the difference of the corresponding elements of two different rows remains constant so that the vectors formed from the rows corresponding to e_i and e_j consist of all vectors of the class corresponding to e_k. Since in our scheme among the differences of corresponding elements of any two rows, each element of M occurs just λ times, when our scheme is expanded and each element replaced by the corresponding row of the addition table, every vector will occur λ times. (Replacing the elements by the corresponding suffixes will change the set Σ from M to the set 0, 1, 2, $\cdots$, $s - 1$.)

6. Construction of a completely resolvable array $(\lambda s^2, \lambda s, s, 2)$ of strength 2 and λs constraints, when the index λ and the number of levels s are both powers of a prime p. Let $\lambda = p^u$, $s = p^v$. Consider the Galois field $GF(p^{u+v})$. The elements of the field can be expressed either as powers x^i of a primitive element $x(i = 0, 1, \cdots, p^{u+v} - 1)$ together with the element zero, or as polynomials of degree $u + v - 1$ with coefficient from $GF(p)$, the field of residue classes (mod p). (For a brief exposition of these properties of Galois fields see [11] and [12].) To add two elements we use the polynomial form adding the coefficients (mod p), and to multiply we use the power form remembering the relation

$$x^{p^{u+v}} = x. \tag{6.0}$$

For example if $p = 2$, $u = 1$, $v = 2$, we consider the Galois field $GF(2^3)$, whose elements may be exhibited (using the minimum function $x^3 + x^2 + 1$) as

$$\begin{aligned}
\alpha_0 &= 0 = 0\\
\alpha_1 &= 1 = x^0,\\
\alpha_2 &= x = x,\\
\alpha_3 &= x + 1 = x^5,\\
\alpha_4 &= x^2 = x^2,\\
\alpha_5 &= x^2 + 1 = x^3,\\
\alpha_6 &= x^2 + x = x^6,\\
\alpha_7 &= x^2 + x + 1 = x^4.
\end{aligned} \tag{6.1}$$

We have ordered the elements of the field in what may be called the lexicographic order, that is, if $\alpha_i = a_2x^2 + a_1x + a_0$ then the integer i is expressed as $a_2a_1a_0$ in the scale of numeration with radix 2. The same is done for the general case GF (p^{u+v}). If

$$\alpha_i = a_{n-1}x^{n-1} + \cdots + a_vx^v + a_{v-1}x^{v-1} + \cdots + a_1x + a_0, \tag{6.2}$$

then $i = a_{n-1} \cdots a_1a_0$ in the scale of numeration radix p, where $n = u + v$.

Consider the sub-class M of the elements of $GF(p^{u+v})$ for which the coefficients of x^v and higher powers of x are zero, when the element is expressed in the polynomial form. In our example the sub-class M consists of the elements α_0, α_1, α_2, α_3. In general M will consist of the first p^v elements of $GF(p^{u+v})$ when they are arranged in the lexicographic order. We now establish a correspondence between the elements of the field, and the elements of M in the following manner: The element α_i of $GF(p^{u+v})$ given by (6.2) corresponds to the element

$$\alpha_j = a_{v-1}x^{v-1} + \cdots + a_1x + a_0 \tag{6.3}$$

of M, the coefficients of x^{v-1} and lower powers of x for α_j being the same as the coefficients of the corresponding powers of x in α_i. It is clear that α_j is uniquely determined by α_i, and that

$$j = i \pmod{p^v}, \qquad 0 \leqq j < p^v.$$

Conversely to each α_j of M there correspond p^u elements of $GF(p^{u+v})$, since if α_j is given by (6.3) then for α_i the coefficients $a_{p-1}, \cdots, a_v$ are arbitrary each taking p possible values. It should be noticed that M is a direct factor module in $GF(p^{u+v})$ and that the correspondence used by us is a projection.

In the example under consideration the correspondence between the elements of $GF(2^3)$ and M is given by

$$\begin{aligned} &\alpha_4, \alpha_0 \rightarrow \alpha_0, \\ &\alpha_5, \alpha_1 \rightarrow \alpha_1, \\ &\alpha_6, \alpha_2 \rightarrow \alpha_2, \\ &\alpha_7, \alpha_3 \rightarrow \alpha_3. \end{aligned} \tag{6.4}$$

If we write down the rows of the multiplication table of $GF(p^{u+v})$ and then replace each element by the corresponding element in M, we get a p^{u+v} rowed scheme which can be shown to satisfy the conditions of Theorem 3. If we take the difference of the corresponding elements in any two rows of the multiplication table, then every element of $GF(p^{u+v})$ occurs exactly once. Also if the elements α_i, $\alpha_{i'}$ of the field correspond to the elements of α_j, $\alpha_{j'}$ of M, then the element $\alpha_i - \alpha_{i'}$ of the field corresponds to the element $\alpha_j - \alpha_{j'}$ of M. This shows that in the scheme we have obtained each element of M occurs exactly $\lambda = p^u$ times, among the differences of the corresponding elements of any two rows. It follows from Theorem 3, that if each element of the scheme is now replaced by the corresponding row of the addition table of M (retaining only the suffixes) we get the completely resolvable array $(\lambda s^2, \lambda s, s, 2)$ where $\lambda = p^u$, $s = p^v$.

For example when $p = 2$, $u = 1$, $v = 2$, we have to write down the rows of the multiplication table of $GF(2^3)$. This can be done by using the identifications given in (6.1), remembering that $x^8 = x$. We thus get

$$
(6.5) \qquad
\begin{array}{cccccccc}
\alpha_0 & \alpha_0 & \alpha_0 & \alpha_0 & \alpha_0 & \alpha_0 & \alpha_0 & \alpha_0 \\
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\
\alpha_0 & \alpha_2 & \alpha_4 & \alpha_6 & \alpha_5 & \alpha_7 & \alpha_1 & \alpha_3 \\
\alpha_0 & \alpha_3 & \alpha_6 & \alpha_5 & \alpha_1 & \alpha_2 & \alpha_7 & \alpha_4 \\
\alpha_0 & \alpha_4 & \alpha_5 & \alpha_1 & \alpha_7 & \alpha_3 & \alpha_2 & \alpha_6 \\
\alpha_0 & \alpha_5 & \alpha_7 & \alpha_2 & \alpha_3 & \alpha_6 & \alpha_4 & \alpha_1 \\
\alpha_0 & \alpha_6 & \alpha_1 & \alpha_7 & \alpha_2 & \alpha_4 & \alpha_3 & \alpha_5 \\
\alpha_0 & \alpha_7 & \alpha_3 & \alpha_4 & \alpha_6 & \alpha_1 & \alpha_5 & \alpha_2 .
\end{array}
$$

Using the correspondence (6.4) the difference scheme is given by

$$
(6.6) \qquad
\begin{array}{cccccccc}
\alpha_0 & \alpha_0 & \alpha_0 & \alpha_0 & \alpha_0 & \alpha_0 & \alpha_0 & \alpha_0 \\
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_0 & \alpha_2 & \alpha_0 & \alpha_2 & \alpha_1 & \alpha_3 & \alpha_1 & \alpha_3 \\
\alpha_0 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 \\
\alpha_0 & \alpha_0 & \alpha_1 & \alpha_1 & \alpha_3 & \alpha_3 & \alpha_2 & \alpha_2 \\
\alpha_0 & \alpha_1 & \alpha_3 & \alpha_2 & \alpha_3 & \alpha_2 & \alpha_0 & \alpha_1 \\
\alpha_0 & \alpha_2 & \alpha_1 & \alpha_3 & \alpha_2 & \alpha_0 & \alpha_3 & \alpha_1 \\
\alpha_0 & \alpha_3 & \alpha_3 & \alpha_0 & \alpha_2 & \alpha_1 & \alpha_1 & \alpha_2 .
\end{array}
$$

To obtain the completely resolvable array (32, 8, 4, 2) we replace the α's in (6.6) by the suffixes in the addition table of M. These replacements are

$$
(6.7) \qquad
\begin{array}{l}
\alpha_0 \to 0,\ 1,\ 2,\ 3, \\
\alpha_1 \to 1,\ 0,\ 3,\ 2, \\
\alpha_2 \to 2,\ 3,\ 0,\ 1, \\
\alpha_3 \to 3,\ 2,\ 1,\ 0.
\end{array}
$$

Finally the orthogonal array (32, 9, 4, 2) can be obtained by adding a final row consisting successively of 8 zeros, 8 ones, 8 twos and 8 threes. The completed array is

$$
(6.9) \qquad
\begin{array}{l}
0\ 1\ 2\ 3\ 0\ 1\ 2\ 3\ 0\ 1\ 2\ 3\ 0\ 1\ 2\ 3\ 0\ 1\ 2\ 3\ 0\ 1\ 2\ 3\ 0\ 1\ 2\ 3\ 0\ 1\ 2\ 3 \\
0\ 1\ 2\ 3\ 1\ 0\ 3\ 2\ 2\ 3\ 0\ 1\ 3\ 2\ 1\ 0\ 0\ 1\ 2\ 3\ 1\ 0\ 3\ 2\ 2\ 3\ 0\ 1\ 3\ 2\ 1\ 0 \\
0\ 1\ 2\ 3\ 2\ 3\ 0\ 1\ 0\ 1\ 2\ 3\ 2\ 3\ 0\ 1\ 1\ 0\ 3\ 2\ 3\ 2\ 1\ 0\ 1\ 0\ 3\ 2\ 3\ 2\ 1\ 0 \\
0\ 1\ 2\ 3\ 3\ 2\ 1\ 0\ 2\ 3\ 0\ 1\ 1\ 0\ 3\ 2\ 1\ 0\ 3\ 2\ 2\ 3\ 0\ 1\ 3\ 2\ 1\ 0\ 0\ 1\ 2\ 3 \\
0\ 1\ 2\ 3\ 0\ 1\ 2\ 3\ 1\ 0\ 3\ 2\ 1\ 0\ 3\ 2\ 3\ 2\ 1\ 0\ 3\ 2\ 1\ 0\ 2\ 3\ 0\ 1\ 2\ 3\ 0\ 1 \\
0\ 1\ 2\ 3\ 1\ 0\ 3\ 2\ 3\ 2\ 1\ 0\ 2\ 3\ 0\ 1\ 3\ 2\ 1\ 0\ 2\ 3\ 0\ 1\ 0\ 1\ 2\ 3\ 1\ 0\ 3\ 2 \\
0\ 1\ 2\ 3\ 2\ 3\ 0\ 1\ 1\ 0\ 3\ 2\ 3\ 2\ 1\ 0\ 2\ 3\ 0\ 1\ 0\ 1\ 2\ 3\ 3\ 2\ 1\ 0\ 1\ 0\ 3\ 2 \\
0\ 1\ 2\ 3\ 3\ 2\ 1\ 0\ 3\ 2\ 1\ 0\ 0\ 1\ 2\ 3\ 2\ 3\ 0\ 1\ 1\ 0\ 3\ 2\ 1\ 0\ 3\ 2\ 2\ 3\ 0\ 1 \\
0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3.
\end{array}
$$

It should be noted that 9 is the maximum possible number of constraints for an orthogonal array of size 32 and strength 2 with 4 levels (cf. Theorem 1B).

7. Adjunction of new rows to the completely resolvable array (λs^2, λs, s, 2), where $\lambda = p^u$, $s = p^v$ and p is a prime. As already explained we can add at least one more row to the array without destroying its orthogonality giving $\lambda s + 1$ constraints in all. Let

$$u = cv + d, \qquad c \geqq 0, \quad 0 \leqq d < v.$$

If $c = 0$, we stop after one row has been added. But if $c > 0$ we shall show that we can do better. Now $u \geqq v$. Let us denote by (A_0) the original completely resolvable array $(\lambda s^2, \lambda s, s, 2)$. Using the same construction as for (A_0), we can obtain another completely resolvable array $(\lambda_1 s^2, \lambda_1 s, s, 2)$ where $\lambda_1 = p^{u-v}$. Let us call this array (A_1). It should be noticed that the number of columns in (A_1) is equal to the number of arrays of strength unity composing (A_0) since $\lambda_1 s^2 = \lambda s = p^{u+v}$. We now inflate (A_1) by repeating each column s times, thus arriving at the array (A_1'), which has the same number of columns as (A_0). We now adjoin (A_1') to (A_0) placing the former just below (A_0). The result is that below any component of (A_0) we get the same column of (A_1) repeated s times. In view of the resolvability property of (A_0) it is clear that if we choose a particular row of (A_0) and a particular row of (A_1') then every ordered pair occurs λ times. Hence the whole array $\begin{pmatrix} A_0 \\ A_1' \end{pmatrix}$ is of strength 2 and has $\lambda s + \lambda_1 s$ constraints.

Since A_1 is completely resolvable, $\begin{pmatrix} A_0 \\ A_1' \end{pmatrix}$ is s-resolvable. If $c = 1$, then $\lambda_1 < s$, we stop after adjoining a final row to $\begin{pmatrix} A_0 \\ A_1' \end{pmatrix}$ consisting of λs zeros followed by λs ones and so on, getting $\lambda s + \lambda_1 s + 1$ constraints in all.

On the other hand if $c > 1$, we do not adjoin the final row as yet but construct a completely resolvable array $(\lambda_2 s^2, \lambda_2 s, s, 2)$ where $\lambda_2 = p^{u-2v}$. Denote this array by (A_2). We next inflate (A_2) to (A_2'') by repeating each column s^2 times and adjoin it to $\begin{pmatrix} A_0 \\ A_1' \end{pmatrix}$ arriving at the array $\begin{pmatrix} A_0 \\ A_1' \\ A_2'' \end{pmatrix}$ of strength 2 with $\lambda s + \lambda_1 s + \lambda_2 s$ constraints. If $c = 2$ we finish the process by adding the final row but if $c > 2$ we continue on as before.

The whole process therefore leads to an orthogonal array of strength 2 in which the number of constraints is given by

$$\lambda s + \lambda_1 s + \cdots + \lambda_c s + 1, \qquad \lambda_i = \lambda / s^i. \tag{7.0}$$

We can therefore state the following theorem.

THEOREM 4. *Given $s = p^v$, $\lambda = p^u$ (where p is a prime) then we can construct an orthogonal array $(\lambda s^2, k, s, 2)$ of strength 2, in which the number of constraints k is given by*

$$k = \frac{\lambda(s^{c+1} - 1)}{s^c - s^{c-1}} + 1, \tag{7.1}$$

where $c = [u/v]$.

8. The use of finite projective geometries in the construction of orthogonal arrays.

THEOREM 5A. *If we can find a matrix C of k rows and r columns*

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1r} \\ c_{21} & c_{22} & \cdots & c_{2r} \\ \vdots & \vdots & & \vdots \\ c_{kl} & c_{k2} & \cdots & c_{kr} \end{pmatrix} \tag{8.0}$$

whose elements c_{ij} belong to the Galois field GF(p^n), *and for which every partial matrix obtained by taking t rows is of rank t, then we can construct an orthogonal array (s^r, k, s, t), where $s = p^n$.*

PROOF. Consider $r \times 1$ column vectors ξ whose coordinates belong to GF(p^n). Then there are s^r different ξ. Form the matrix A whose s^r columns are the $k \times 1$ vectors $C\xi$. Then A is the required orthogonal array.

If A' is a $t \times s^r$ submatrix of A, and C' is the corresponding $t \times r$ submatrix of C, the columns α of A' are $C'\xi$, and since C' is of rank t, each α is obtained from s^{r-t} different ξ. Hence in A' each possible $t \times 1$ column vector occurs with the frequency $\lambda = s^{r-t}$, which shows that A is an orthogonal array of strength t and index λ.

The rows of the matrix C may be interpreted as the coordinates of a point in a finite projective space PG$(r - 1, p^n)$ such that no t of the points are conjoint. We thus get the following theorem:

THEOREM 5B. *If we can find k points in* PG$(r - 1, p^n)$ *so that no t are conjoint, then we can construct an orthogonal array (s^r, k, s, t) for which $\lambda = s^{r-t}$, $s = p^n$.*

It has been shown by Bose [7] that the maximum number of factors that it is possible to accommodate in a symmetrical factorial experiment in which each factor is at $s = p^n$ levels, and each block is of size s^r, without confounding any t-factor or lower order interaction, is given by the maximum number of points that it is possible to choose in the finite projective space PG$(r - 1, p^n)$ so that no t of the chosen points are conjoint (a set of t points are said to be conjoint if they lie on a flat space of dimensions not greater than $t - 2$). This number is denoted by $m_t(r, s)$. It is clear from Theorem 5B that we can always construct an orthogonal array (s^r, k, s, t), for which the number of constraints $k \leqq m_t(r, s)$, if $s = p^n$ where p is a prime. The value of $m_t(r, s)$ has been determined by Bose in a number of important cases, and the corresponding set of points in which no t are conjoint has been obtained. These results are used in the next section to construct some orthogonal arrays of strength 3.

9. Construction of some orthogonal arrays of strength 3.

(a) Consider the special case $s = 2$. In PG$(r - 1, 2)$ consider the set of all points, which do not lie on the $(r - 2)$-flat

$$x_1 + x_2 + \cdots + x_r = 0. \tag{9.0}$$

There are exactly 2^{r-1} such points, namely the points in whose coordinates there are an odd number of unities, and the rest zero. No three of these points are collinear since in PG$(r - 1, 2)$ each line passes through exactly three points, and one of these lying in the plane (9.0) is excluded from our set. Taking the coordinates of these points for the rows of the matrix C of Theorem 5A, we can construct the orthogonal array $(2^r, 2^{r-1}, 2, 3)$ of strength 3 and 2^{r-1} constraints. Theorem 2A shows that this is the maximum possible number of constraints.

As an illustration consider the case $r = 3$. The four points of PG(2, 2) not lying on the line $x_1 + x_2 + x_3 = 0$ are (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1). Hence the corresponding matrix C is

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{9.1}$$

The eight possible column vectors ξ are

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1. \end{matrix}$$

The columns of the required array (8, 4, 2, 3) are obtained by forming $C\xi$ given below.

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1. \end{matrix} \tag{9.2}$$

Similarly the array (16, 8, 2, 3) is given by (9.3)

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1. \end{matrix} \tag{9.3}$$

(b) Let $s = 2^n$. In the finite projective plane PG$(2, 2^n)$ take the non-degenerate conic

$$ax_1^2 + bx_2^2 + cx_3^2 + fx_2x_3 + gx_3x_1 + hx_1x_2 = 0, \tag{9.4}$$

where

$$\Delta = af^2 + bg^2 + ch^2 + fgh \neq 0.$$

Of course no three of the points $P_1, P_2, \cdots, P_{s+1}$ on the conic (9.4) are collinear since no line can cut it in more than two points. Through any point (x_1', x_2', x_3') on the conic there pass $s + 1$ lines, of which s join it to the remaining points of the conic, while there is one line which does not meet the conic in any point other than (x_1', x_2', x_3'). This may be called the tangent at (x_1', x_2', x_3'). Its equation is

$$f(x_3'x_2 + x_2'x_3) + g(x_1'x_3 + x_3'x_1) + h(x_2'x_1 + x_1'x_2) = 0. \tag{9.5}$$

It is a peculiar feature of the finite projective geometry, based on a field of characteristic 2, that every tangent to a given non-degenerate conic passes through the same point. For example in the present case the arbitrary tangent (9.5), passes through the point P_0 with coordinates (f, g, h). The $s + 1$ tangents to (9.4) account for all the lines which pass through P_0. Hence no line through P_0 can meet the conic in more than one point. Thus $P_0, P_1, P_2, \cdots, P_{s+1}$ is a set of $s + 2$ points, such that no three are collinear. Hence from Theorem 5B we can use the coordinates of these points to construct an orthogonal array $(s^3, s + 2, s, 3)$ where $s = 2^n$.

Similarly when $s = p^n$ where p is an odd prime we could construct the array $(s^3, s + 1, s, 3)$ by using the coordinates of the $s + 1$ points on a non-degenerate conic of PG$(2, p^n)$.

One of the authors, Bush [10], has shown that for an orthogonal array (s^t, k, s, t) of index unity and strength t, the number of constraints k satisfies the inequality

$$k \leqq s + t - 1 \qquad \text{when } s \text{ is even,} \tag{9.6a}$$

$$k \leqq s + t - 2 \qquad \text{when } s \text{ is odd.} \tag{9.6b}$$

Using this result for $t = 3$, we find that the number of constraints obtained by us for arrays of size s^3, s levels and strength 3, cannot be improved.

(c) Let $\phi(x, y) = ax_1^2 + 2hx_1x_2 + bx_2^2$ be a homogeneous quadratic with coefficients belonging to GF(p^n) and irreducible in it. If $s = p^n$, it can be shown [7] that the quadratic surface

$$ax_1^2 + 2hx_1x_2 + bx_2^2 = x_3x_4$$

contains exactly $s^2 + 1$ points no three of which are collinear. We therefore get a method of constructing an orthogonal array $(s^4, k, s, 3)$ with $k = s^2 + 1$ constraints, when s is a prime or a prime power. On the other hand, Theorem 2C gives an upper bound $s^2 + s$ for k when $s \neq 2, 4, 7$ or 16 and an upper bound for k for these exceptional values of s is given as $s^2 + s + 2$ by Theorem 2A. Thus there remains a gap between the number of constraints which might be attainable, and the number of constraints actually attained except for the case $s = 2$, for which we have already obtained an array $(s^4, 8, 2, 3)$ by the method (a). It is not known whether this gap can be bridged. It has been shown [7] that when p is odd we cannot get more than $s^2 + 1$ points in PG$(3, p^n)$ no three of which are collinear. The same has been proved by Seiden [13] for the case $s = 2^2$.

Hence for these cases the geometrical method cannot lead to more than $s^2 + 1$ constraints, but there remains the possibility that some other combinatorial procedure may lead to a larger number of constraints.

As example consider the case $s = 3$. The coordinates of the 10 points lying on the quadric $x_1^2 + x_2^2 = x_3x_4$ of $PG(3, 3)$ are (0, 0, 1, 0), (0, 0, 0, 1), (0, 1, 1, 1), (0, 1, 2, 2), (1, 0, 1, 1), (1, 0, 2, 2), (1, 1, 1, 2), (1, 1, 2, 1), (1, 2, 1, 2), (1, 2, 2, 1). Using these as the rows of the matrix C, we get the orthogonal array (81, 10, 3, 3), and 10 is the maximum number of constraints obtainable by the geometrical methods. Theorem 2C gives $k \leqq 12$. We do not know whether we can get 11 or 12 constraints in any other way.

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