to be subject to certain indeterminate errors." This remark was intended simply to call attention to the discussion of the accuracy of the table made by the authors of [9] and should not discourage the use of the table since the present writer was merely referring to the evaluation of accuracy made by the authors themselves.

Meanwhile Dr. Churchill Eisenhart has pointed out that an exact table for the case of two degrees of freedom ($r = 0$) has been added to reference [1] as printed in *Contributions to Mathematical Statistics* by R. A. Fisher (John Wiley and Sons, 1950).

ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Chicago meeting of the Institute, December 27–30, 1952)

1. A Two-Sample Multiple Decision Procedure for Ranking Means of Normal Populations with Unknown Variances. (Preliminary Report.) ROBERT E. BECHHOFER, CHARLES W. DUNNETT AND MILTON SOBEL, Cornell University.

The multiple decision problem of ranking k normal populations according to their population means when both the means and the variances are unknown is considered. Useful confidence statements, which are independent of the unknown variances, can be made concerning the rankings if a generalized Stein two-sample procedure is used. The rankings are of the general type formulated by Bechhofer (*Ann. Math. Stat.*, Vol. 23 (1952), p. 139), which includes selecting the population with the largest population mean and ranking completely all k populations. Two cases are considered: A) population variances unknown and equal, and B) population variances unknown and not necessarily equal. To determine the size (which may be zero) of the second sample, tables for a $(k - 1)$-variate analogue of Student's t-distribution are required for case A, and tables for a $(k - 1)$-variate analogue of the distribution of the difference between two independent Student t-statistics are required for case B. For $k = 2$, tables due to Fisher and Sukhatme are available for cases A and B, respectively. For case A, $k = 3$, tables are being computed. For case A, $k > 3$ and for case B, $k \geqq 3$, computation of the necessary tables is contemplated. (This research was sponsored by Air Research and Development Command.)

2. A Sequential Multiple Decision Procedure for Ranking Means of Normal Populations with Known Variances. (Preliminary Report.) ROBERT E. BECHHOFER AND MILTON SOBEL, Cornell University.

Let X_{ij} be normally and independently distributed $N(X_{ij} \mid \mu_i, \sigma^2)$ from population π_i $(i = 1, \cdots, k; j = 1, 2, \cdots)$. The μ_i are unknown; σ^2 is known. Denote the ranked μ_i by $\mu_{[1]} < \cdots < \mu_{[k]}$. It is desired to choose the population with mean $\mu_{[k]}$. Denote the sample sum based on m observations from π_i by $Q_{im} = \Sigma_{j=1}^{m} x_{ij}$; denote the ranked Q_{im} by $R_{1m} < \cdots < R_{km}$. Define $y_m = R_{km} - R_{k-1,m}$ $(m = 1, 2, \cdots)$ and $\delta_{k,i} = \mu_{[k]} - \mu_{[i]}$ $(i = 1, \cdots, k - 1)$. Let $\delta^* > 0$ be the smallest value of $\delta_{k,k-1}$ that it is desired to detect, and let $1 - \beta(1/k < 1 - \beta < 1)$ be the desired probability of a correct choice (p.c.c.)

when $\delta_{k,k-1} \geqq \delta^*$. Let $C = (k-1)(1-\beta)/\beta$. The sequential procedure S_D given below, where $D = (\sigma^2/\delta^*) \log_e C$, has the property that: p.c.c. is $\geqq 1 - \beta$ when $\delta_{k,k-1} \geqq \delta^*$ with equality holding when $\delta_{k,i} = \delta^*$ $(i = 1, 2, \cdots, k-1)$. The procedure S_D is: "Observe the k-tuple $(x_{11}, \cdots, x_{k1})$ and compute y_1. If $y_1 \geqq D$, cease taking observations and choose the population associated with R_{k1} as the one having mean $\mu_{[k]}$. If $y_1 < D$, observe the k-tuple $(x_{21}, \cdots, x_{2k})$ and compute y_2. Proceed as above for $m = 2, 3, \cdots$ until observation-taking ceases and a population is chosen." When S_D is used, the g.l.b. of the p.c.c. (neglecting, as in Wald-sequential, the excess over the boundary) is given by $L(\delta) = C^{\delta/\delta^*} \div \{C^{\delta/\delta^*} + (k-1)\}$ for all $\delta_{k,i}$ for which $\delta = \text{Min } \delta_{k,i}$ $(i = 1, \cdots, k-1)$. Truncation of S_D also has been studied. Generalizations of S_D solve a large class of ranking problems. (Research sponsored by Air Research and Development Command.)

3. On Quadratic Estimates of Variance Components. FRANKLIN GRAYBILL, Oklahoma A. & M. College and Iowa State College.

A common method of estimating variance components is "the analysis of variance" method where the estimate is obtained by equating expected mean squares to observed mean squares in an analysis of variance table and solving the resulting equations for the desired variance components. For any infinite population let the general hierarchal classification model be given by $y_{ij\ldots p} = \mu + a_i + b_{ij} + \cdots + e_{ij\ldots p}$ where μ is a constant and $a_i, b_{ij}, \cdots, e_{ij\ldots p}$ are independent random variables with means zero, variances σ_1^2, $\sigma_2^2, \cdots, \sigma_p^2$ respectively, and with finite fourth moments. Let $\hat{\sigma}_k^2$ be "the analysis of variance estimate" for σ_k^2. It is shown that the quadratic estimate of $\Sigma_{k=1}^p g_k\sigma_k^2$ (g_k known) which is unbiased, independent of μ, and has minimum variance is given by $\Sigma_{k=1}^p g_k\hat{\sigma}_k^2$. That is to say, the best unbiased quadratic estimate of a linear combination of variance components is given by the same linear combination of "the analysis of variance estimates" of the individual variance components.

4. Topics in Analysis of Variance: A. Optimum Properties of Tests for Model II, B. Generalizations of Model II. (Preliminary Report.) LEON HERBACH, Brooklyn College and Columbia University.

A canonical form analogous to the one used for the general linear hypothesis is developed for Model II analysis of variance for one-way classifications and balanced (in the sense of Crump, *Biometrics*, Vol 7 (1951), pp. 1–16), multiple classifications. From this it is shown that 1) all exact F-tests used in testing hypotheses based on balanced multiple classifications determine uniformly most powerful (u. m. p.) similar regions although they are not always likelihood ratio tests; 2) in the case of one-way classifications the F-test is a u. m. p. invariant test; if the classification is unbalanced there exists no u. m. p. similar region; if it is balanced the F-test is a likelihood ratio test. Two new models are considered which do away with the objections usually made to Model II, namely that 1) in multiple classifications the interactions must be distributed independently of the main effects and 2) one cannot account for a larger frequency of negative estimates of variance components than can be explained by sampling errors. These models have as special cases the usual Model II. (Work partially sponsored by the Office of Naval Research.)

5. Further Examples for which the Likelihood Ratio Test is Poor. L. M. COURT, Brooklyn College.

In an earlier paper, the writer gave a very general method for constructing situations in which the likelihood ratio was a poor test. That is, very general families of density distributions were constructed and hypotheses, both simple and composite, set up within these families such that the probability of rejecting any element of the hypothesis was greater when that element was true than when any element of the alternative was true. This was so with reference to a suitable critical region that was part of the construction. (Previously both Rubin and Stein had given simple, restricted examples in which the likelihood ratio test was poor.) In these constructions, the dimensionality of the parameter space was always the *same* as the dimensionality of the variate space. In the present paper, the writer shows how by building upon the earlier constructions and using product spaces, it is possible to give examples for which the likelihood ratio test is poor and these two dimensionalities are *different*.

6. **An Empirical Sampling Method.** E. L. Cox, Dugway Proving Ground.

Certain sequences of dependent trials where there are at least two possible outcomes at each trial give rise to such probability distributions as

$$P(X = t) = [(N - T)_{n-t}/N^n][\Delta^{n-t}\, T^n/(n - t)!], \quad \text{or } P(X = n) = [(N - T)_{n-t}/N^n](T + n - t)[\Delta^{n-t}\, T^{n-1}/(n - t)!],$$

etc., where N and n are population size and sample size respectively and T and t are the number of individuals possessing a certain attribute in the population and sample respectively. The individual probabilities from such distributions may be difficult to evaluate explicitly. A method using IBM punched cards has been devised to approximate the probability distributions. A set of random numbers X are punched from a deck containing a random number table. These random numbers are transformed by machine methods to the desired random number set Y under the relation $X \equiv Y \pmod{N}$. The random variables Y are used in the machine operations from which the probability approximations can be evaluated. Some worked examples are shown.

7. **Distribution of Semi-Definite and of Indefinite Quadratic Forms.** John Gurland, Iowa State College.

The problem considered is that of finding the distribution function of $\Sigma_1^n \lambda_i X_i^2$ where $X_1, X_2, \cdots, X_n$ are independent normal random variables each with mean zero and unit variance. The method used to solve this problem extends a result of Bhattacharya (*Sankhya*, 1945). A convenient expansion of the characteristic function and then an application of the inversion formula yields a series in Laguerre polynomials which converges to the exact distribution. For definite or semi-definite forms, this expansion is a special case of previous results (submitted to the *Annals of Mathematical Statistics*) obtained by the author. For indefinite forms, the series obtained applies to the case where the number of positive λ's or the number of negative λ's is even. There are no other restrictions on the λ's.

8. **Asymptotic Properties of Ideal Linear Estimators.** Carl Allen Bennett General Electric Company.

The problem studied is the determination of the asymptotic properties of ideal linear estimators of the location and scale parameters of a continuous univariate distribution function. An ideal linear estimator is defined as a linear combination of the order statistics which provides an unbiased estimate of a particular parameter with minimum variance. It is shown that if the probability density function is continuous and nonvanishing over the range of possible values, then the ideal linear estimators of the location and scale parameters based on all the order statistics are asymptotically efficient. The limiting form of the coefficients of these estimators is obtained. The final results are obtained in a sufficiently general form so that they can be applied to truncated samples and the effect of such truncation studied. Certain situations in which the probability density function is discontinuous and the estimators "super-efficient" are dealt with as limiting cases of a truncated sample. (This paper summarizes the principal developments in a thesis of the same title submitted to the University of Michigan in February 1952 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.)

9. **Bias in Estimation by Interval.** M. C. K. TWEEDIE, Introduced by B. HARSHBARGER, Virginia Polytechnic Institute.

Suppose that a sample of n values is drawn from a constant univariate population Π (not necessarily continuous). It is well known (cf. K. R. Nair, 1940) that the closed interval extending from the rth smallest to the rth largest observation can, with an appropriate choice of r, be used as a confidence interval for the median of Π. Under a condition which is nearly always satisfied in practice, no quantile of Π is more likely to be covered by this interval than is the median. Apart from the slight restriction referred to, this seems to be a very satisfactory property. In general, a confidence interval might be more likely to cover some incorrect value of the parameter than to cover the true value, and it would then represent a case of "bias" in a sense which appears not to have been discussed previously. Let $A(\theta \mid \theta')$ denote the probability that a confidence interval will include θ when θ' is the true value. It is proposed that the dependence of $A(\theta \mid \theta')$ on θ, at various constant θ', should be considered in setting up definitions of desirable properties of confidence intervals. This idea is illustrated by a number of examples. Neyman's (1937) discussion of "unbiassed" systems of confidence intervals is based on the dependence of $A(\theta \mid \theta')$ on θ' at constant θ, which is not necessarily equivalent to the new proposal.

10. **Completeness of the Order Statistics in the Nonparametric Case.** D. A. S. FRASER, University of Toronto.

In sampling from an unknown distribution on the real line, the order statistics are known to be sufficient. Here they are shown to form a complete statistic for any class of absolutely continuous distributions which contains at least all uniform distributions over finite numbers of intervals. In $U\,M\,V$ unbiased estimation in the nonparametric case, the Lehmann-Scheffé theorem remains valid so long as the order statistics remain complete for these distributions for which the parameter exists. Any estimable parameter which exists for bounded distributions has then a $U\,M\,V$ unbiased estimate.

11. **Parameter-free and Nonparametric Tolerance Limits: The Exponential Case.** LEO A. GOODMAN, University of Chicago.

Parameter-free tolerance limits are developed based on the first r ordered observations $x_1 \leqq x_2 \leqq \cdots \leqq x_r$ from a sample of n exponentially distributed variates. Comparisons between these tolerance limits and nonparametric limits are made. For example, although $E\{x_1\} = E\{\sum_{i=1}^{n} x_i/n^2\}$, we find that a sample of 459 observations is required to obtain 99 per cent tolerance limits at probability level 99 per cent if the limits are the nonparametric (x_1, ∞); and only 122 observations are required if the limits are the parameter-free $(\sum_{i=1}^{n} x_i/n^2, \infty)$. The exact behavior of the coverage is investigated; for example, if the parameter-free limits are of the form $([\sum_{i=1}^{r} x_i + (n - r)x_r]c/r, \infty)$, where c is a constant, the distribution of the coverage is obtained. The asymptotic behavior of the coverage is also studied. It is shown that the parameter-free limits are asymptotically better than the nonparametric limits (x_i, ∞) whenever $1 - cn/r > \beta$ for 100β per cent tolerance limits. Tables of numerical comparisons are made. The method of obtaining parameter-free tolerance limits may be generalized to the case where the distribution of the observations has the invariance property under change of scale.

12. On Estimates Whose Distributions Have a Weak Invariance Property. LEO A. GOODMAN, University of Chicago.

Let X be a real-valued random variable whose distribution depends on an unknown real parameter θ. Suppose $\theta E\{X \mid \theta\}/E\{X^2 \mid \theta\} = A$, where A is known. Then among all statistics of the form aX, where a is a constant, the mean square error $E\{(aX - \theta)^2 \mid \theta\}$ is minimized uniformly when $a = A$. Also, among all statistics of the form aX, the only one which is unbiased with respect to the loss function $W(\theta, f(X)) = (f(X) - \theta)^2/\theta^2$ is AX, the statistic which minimized the mean square error. As a corollary of this result we have the following. Suppose X is an unbiased estimate (in the usual sense) of θ whose coefficient of variation is a known constant V. Then the estimate $X/(V^2 + 1)$ has a smaller mean square error than the estimate X. The relative improvement in the mean square error by using $X/(V^2 + 1)$, rather than X, is $V^2/(V^2 + 1)$. The estimate $X/(V^2 + 1)$ is unbiased with respect to the loss function $W(\theta, f(X))$. Another corollary of this result is related to the problem of the estimation of the scale parameter of a population whose form may not be known but the ratio of the first and second moments is given. Applications of this result are presented which are related to the problem of estimating the standard deviation and variance of normal variates, the range of uniform variates, and other problems.

13. The Admissibility of Certain Statistical Tests. (Preliminary Report.) ERICH L. LEHMANN AND CHARLES M. STEIN, University of California, Berkeley and University of Chicago.

Let ω_0, ω_1 be one-parameter families of probability measures, each element of ω_i being obtained from a single element of ω_i by a translation. Under weak restrictions, the unique most powerful test of given size among all those invariant under translation is admissible in the class of all tests. This is true in particular for the simplest applications of (central or noncentral) Student's t test: $X_1, \cdots, X_n$ are independently normally distributed with unknown mean ξ and unknown variance σ^2. An admissible test for $H_0 : \xi/\sigma = \delta_0$ against $H_1 : \xi/\sigma = \delta_1 > \delta_0$ (where δ_0, δ_1 are given numbers) is given by: Reject H_0 if $\Sigma X_i \geqq k\sqrt{\Sigma X_i^2}$.

14. Some Experimental Designs for Comparative Life-Testing. ALLAN BIRNBAUM, Columbia University.

Consider two populations of objects whose "durations of life" t have respective densities $f(t, \theta_i) = (1/\theta_i) \exp[-t/\theta_i]$, $i = 1, 2$. Let M objects from each population be observed continuously, with each item failing immediately being replaced by another from the same population. Let $Y_j = 1$ or 0 as the jth failing item is from the first or second population. Let $p = \text{Prob}\{Y_j = 1\}$ and $\gamma = \theta_1/\theta_2$. Then $p = 1/1 + \gamma$. The sequence of observed values $y_1, y_2, \cdots$, may thus be used in sequential or nonsequential procedures to give tests or interval estimates of θ_1/θ_2. M may be altered during experimentation without invalidating these procedures. Observed "waiting times" t_j between observations y_{j-1} and y_j may be used to give independent confidence statements concerning min (θ_1, θ_2) or max (θ_1, θ_2).

15. Existence of Invariant Minimax Procedures. Herman Rubin, Stanford University.

An unpublished result of Hunt and Stein states that, under certain specified conditions, if there is a most stringent test, then there is a most stringent invariant test. This theorem has been translated into a theorem about decision functions and, by a method similar to theirs, it is shown that, under certain restrictions, given any statistical procedure, there is an invariant procedure whose "maximum" risk is no greater than the "maximum" risk of the given procedure. Thus, under those conditions, one may restrict a search for minimax procedures to invariant procedures. A similar though somewhat less general result has been obtained by Peisakoff.

16. A Simple Sequential Procedure for Testing Statistical Hypotheses. (Preliminary Report.) Chia Kuei Tsao, Wayne University.

In applying Wald's sequential probability ratio test, one has to choose a definite alternative hypothesis which may restrict its application. In this paper, a different simple sequential test procedure is suggested. In testing a simple hypothesis, say $f(x) = f_0(x)$, one needs only to divide the whole space into three mutually exclusive and exhaustive zones: the zone of acceptance R_1, the zone of indifference R_2 and the zone of rejection R_3. Random observations are drawn successively. At each stage, the number of observations falling in each of these three zones will be counted. Denote by n_i the number of observations falling in the ith zone ($i = 1, 2, 3$) after the nth observation has been drawn and let $\Sigma n_i = n$. Continue to draw observations as long as $n_1 < k$ and $n_3 < k$ (where k is a positive integer). The experiment is discontinued as soon as $\max (n_1, n_3) = k$. The null hypothesis is accepted if $n_1 = k$ or rejected if $n_3 = k$. Distribution of the sample size, the m. g. f. and the best critical region are obtained. It is also shown that these tests are consistent. Uniformly most powerful and unbiased tests are discussed. For properly chosen zones, they are, on the average, more efficient than some of the standard fixed sample tests. Furthermore, these tests can also be applied to the nonparametric case. (This work was supported in part by the Office of Naval Research.)

17. Asymptotic Properties of the Robbins-Monro Process. Joseph L. Hodges, Jr. and Erich L. Lehmann, University of California, Berkeley.

Robbins and Monro have proposed a class of stochastic processes which converge (in the sense of mean square) to the root of a regression equation. The rate of convergence of the mean square errors of these processes is shown to depend only on the local properties of the model at the root, and thus the asymptotic optimum problem may be solved by considering the easy problem of linear regression with constant variance. Asymptotically, the best Robbins-Monro scheme cannot be improved. As an application, an asymptotically optimum solution of the bio-assay problem is provided for any percentile.

18. Some Notes on the Application of Sequential Methods in the Analysis of Variance. N. L. Johnson, University of North Carolina.

This paper is concerned with points of detail arising in the application of sequential methods to problems which, if fixed sample sizes were used, would be suitable for analysis of variance techniques. Using results due to G. A. Barnard and D. R. Cox it is shown that, in the case of systematic models, a valid sequential test (not dependent on arbitrary weighting functions) may be based on the sequence of deviance ratios calculated at various stages as the experimental design is built up. Similar procedures, in the case of random models, do not always terminate with probability one. Alternative forms of design, such that this difficulty will not arise, are suggested and a short table of critical limits for deviance ratios is provided. (This research was supported by the United States Air Force under Contract AF18(600)-83.)

19. The Covariances of Frequencies from a Multinomial Distribution under a Sequential Sampling Rule.

M. C. K. Tweedie, Introduced by B. Harshbarger, Virginia Polytechnic Institute.

Suppose that a constant multinomial distribution is sampled repeatedly, with replacement, only until some non decreasing linear function Z of the frequencies reaches some prescribed positive value ζ. That is, if x_i is the observed frequency in group i ($i = 1$ to N), and $Z = \Sigma_{i=1}^{N}(\lambda_i x_i)$, where each λ_i is a nonnegative real constant, sampling is stopped as soon as $Z \geqq \zeta$. If π_i is the probability that any one observation will fall in group i, the expectation of the final sample size n is $E(n) \sim \zeta/\Sigma_1^N(\lambda_i\pi_i)$, and $E(x_i) = \pi_i E(n)$, (cf. Blackwell). Wald's fundamental identity can be generalized to $E\{e^{-U}(\Sigma_1^N \pi_i e^{-t_i})^{-n}\} = 1$, where $U = \Sigma_1^N (x_i t_i)$, and (assuming that the final Z exceeds ζ by a negligible amount) it can be shown to follow that cov $(x_i, x_j) = [\zeta\pi_i/\Sigma(\lambda\pi)]\{[\pi_j\Sigma(\lambda^2\pi)/(\Sigma\lambda\pi)^2] - \pi_j(\lambda_i + \lambda_j)/\Sigma(\lambda\pi)] + \delta_{ij}\}$, where $\delta_{ij} = 0$ if $j \neq i$, $\delta_{ii} = 1$. By changing the values of the constants (or scores) $\lambda_1, \lambda_2, \cdots, \lambda_N$, the statistical properties of estimators and other statistics based on samples thus obtained can be varied, and can sometimes be made to approximate to various desiderata.

20. The Totality of Transformations Leaving a Family of Normal Distributions Invariant. (Preliminary Report.)

Erich L. Lehmann and Charles M. Stein, University of California, Berkeley and University of Chicago.

If X is a finite dimensional real linear space, X^* its adjoint, $\Theta \subset X^*$, and μ a σ-finite measure in X, a set of densities $\psi(\theta) \exp(\theta x)$ for $\theta \epsilon \Theta$ is called an exponential family. If f is a 1-1 function on X onto another linear space X' taking the exponential family into another, (Θ', μ') then the induced function $\bar{f}: \Theta \rightarrow \Theta'$ can be extended uniquely to a 1-1 affine function φ on the affine space T generated by Θ onto T'. If L is the linear space parallel to T, then the projection of f on L^* is almost linear, in fact it coincides almost everywhere with the inverse of the adjoint of the linear function on L determined by φ. The invariants of φ are obtained. These considerations are specialized to normal distributions, but the resulting algebraic problem has been solved only for special cases. There are $\bar{f}$'s which cannot be generated by an affine transformation of the normal random vector.

21. A Nonparametric Two-Sample Life Test.

Benjamin Epstein, Wayne University.

Let two samples S_1 and S_2 each of size n be put on a life test. For any fixed m define the random variable Z_m, where Z_m is the number of trials when at least m failures have occurred in both S_1 and S_2 for the first time. Then the rule $Z_m > C_m$, a sufficiently large integer $\geqq 2m$, gives a procedure for rejecting the null hypothesis that S_1 and S_2 are drawn from the same population. The power, expected number of observations to make a decision, and other pertinent features of this test are being investigated experimentally for the case of normal slippage for $n = 2(1)20$ and $m = 1, 2, 3$. The work for $m = 1$ is related to recent slippage tests by Mosteller and Tukey. (Work sponsored by the Office of Naval Research.)

22. The Large Sample Power of a Test Based on Dichotomization. CHIA KUEI TSAO, Wayne University.

Let π be a population having p. d. f. $f(x)$ over a region R. Let R be divided into two mutually exclusive and exhaustive zones R_1 and R_2. Suppose a random sample of size n is drawn from π. Let S be the number of observations of the sample falling in the zone R_1. Then the distribution of the random variable S is given by the binomial distribution: $b(s) = \binom{n}{s} p^s q^{n-s}$, where $p = S_{R_1} f(x)\, dx$. By the use of this binomial distribution, one can test the hypothesis $f(x) = f_0(x)$ against an alternative $f(x) = f_1(x)$. In order to obtain the maximum power for fixed n, one needs to determine an optimum zone R_1. In this paper, it is shown that, for large n, the optimum zone R_1 is one which consists of all those points x where $f_1(x)/f_0(x) \geqq 1$. (Supported in part by the Office of Naval Research.)

23. An ϵ-Complete Class of Nonsequential Decision Procedures in the Finite Case. LIONEL WEISS, University of Virginia.

Suppose there is a finite set of possible decisions: $d_1, d_2, \cdots, d_k$, and a finite number of possible distributions for the vector chance variable X, with density functions $f(x:1)$, $f(x:2), \cdots, f(x:n)$. Then, given any positive ϵ, by a simple use of the Dantzig-Wald converse of the Neyman-Pearson lemma it can be shown that the following class of decision procedures is ϵ-complete: Either d_j or d_i is rejected according to whether or not

$$f(x:s[i,j]) > \Sigma a(r:i,j) f(x:r),$$

where the summation is with respect to r and extends over all r not equal to $s[i, j]$. This sort of comparison is made $k - 1$ times, until only one decision is left, which is the decision finally chosen. The $a(r:i, j)$ are predetermined constants which will in general depend on which decisions have already been rejected by the preceding comparisons. The $s[i, j]$ are also predetermined. In the case where an invariance condition of the following form is imposed: a set of points $T(X)$ containing X is associated with each X and the same decision must be made for any point in the set $T(X)$, then if we replace $f(x:i)$ by $sf(t:i)\, dt$, where the integration extends over $T(X)$, the decision procedures defined in the same way as above in terms of these new functions form an ϵ-complete class among all procedures with the invariance property.

24. Approximate Distribution of Extreme Values of the Range. M. H. BELZ AND ROBERT HOOKE, University of Melbourne and Princeton University.

For a sample range that is considered extreme it might be expected that the order statistics that define it would be approximately independent of each other. Further, a consideration of the cumulative distribution function for the range in samples from a normal population shows that, for extreme values, the distribution is closely normal. These features suggest: (i) the investigation of the distribution of the range under the assumption of independence, and a comparison of the probability obtained from this distribution with that obtained from the exact distribution when the range is extreme; (ii) the development of a practical procedure for approximating to critical probabilities associated with the range, based on the assumption that the order statistics in question are normally distributed around their expected values with characteristic variances. Relations have been obtained between probabilities computed in the above ways for a variety of distribution functions. Some interesting limiting results have also been examined for the ratio of the probability that the range will exceed a given value to that found under the assumption of independence, merely, as the given value increases. As a practical measure, a good approximation to the exact probability that the range will exceed an extreme value appears to be obtained by adopting the normal procedure mentioned, with the required low moments estimated from the data of a number of independent samples, provided that an appropriate co-factor, depending on the sample size, is attached to the probability so found. The method can be readily extended to the more general case where a comparison between two linear functions of the order statistics is made the basis of a statistical test or other procedure. (Work done under the sponsorship of the U. S. Office of Naval Research.)

25. Estimate of the Interval Rate for Actuarial Calculation. (Preliminary Report.) JOSEPH BERKSON, Mayo Clinic.

If the functional form of the survivorship curve is known, a T-year survival rate can be estimated by "fitting" the function to the observations, using some method such as maximum likelihood. If the form is not known, the actuarial method is used. This consists of subdividing the T-year interval into subintervals, estimating the probability of survival in each interval, and obtaining the T-year rate as the product of these. The case is considered in which the individuals of the population concerned are identified, and information concerning death or survival is obtained by periodic survey. It is assumed that in any subinterval the survivorship may be considered linear. Four estimators $\hat{} = 1 - \hat{p}$, the probability of dying in the interval, are considered and compared: (1) $D/[N - W/2]$, (2) $D/[N - L/2]$, (3) the solution of $\hat{q} = D/[N - L/(2 - \hat{q})]$ and (4) the solution of $\hat{q} = D/[N - \Sigma\{L_i(1 - t_i)/(1 - \hat{q}t_i)\}]$ where N is the number observed living at the beginning of the interval, D is the number of observed deaths during the interval, L_i is the number last observed living during the interval at time t_i after the beginning of the interval, $L = \Sigma L_i$ and W is the total number "withdrawn" during the interval.

26. Simultaneous Confidence Interval Estimation. R. C. BOSE AND S. N. ROY, University of North Carolina.

Let $y_i (i = 1, 2, \cdots, N)$ be an observed set of random variables whose joint distribution depends on the unknown parameters $\theta_j (j = 1, 2, \cdots, n)$. Let $\Pi_k = f_k(\theta_1, \theta_2, \cdots, \theta_n)$, $k = 1, 2, \cdots$, be a given finite or infinite set of parametric functions. Suppose it is possible to find a set of functions $\psi_k(y_1, y_2, \cdots, y_N, \Pi_k)$ such that $\psi_k \leqq c$ implies $\phi_{1k}(y_1, y_2, \cdots, y_N) \leqq \Pi_k \leqq \phi_{2k}(y_1, y_2, \cdots y_N)$. Let W be the intersection of the regions $\psi_k \leqq c$ for $k = 1, 2, \cdots$. If Prob $\{(y_1, y_2, \cdots y_N) \varepsilon W\} = 1 - \alpha$ and is independent of the parameters, then $1 - \alpha$ is also the chance that all the statements $\phi_{1k} \leqq \Pi_k \leqq \phi_{2k}$ are simultaneously true. Some results due to Scheffé and Tukey are derived and other examples of simultaneous confidence interval estimation (especially in connection with factorial experiments) are given. Applications to multivariate analysis will be given separately.

27. On a Set of Simultaneous Confidence Interval Statements in Multivariate Analysis of Variance. S. N. ROY AND R. C. BOSE, University of North Carolina.

In an earlier paper (S. N. Roy, "On a heuristic method of test construction and its use in multivariate analysis," to appear in *Ann. Math. Stat.*), among other things, the following test criterion was proposed: Suppose there are k p-variate normal populations with a common covariance matrix Σ and p-dimensional mean column vectors $\underline{\xi}_i(i = 1, \cdots, k)$, and random samples of sizes $n_i + 1$ drawn from them, and let S^* and S stand for the "between" and "within" covariance matrices of the k samples and $n \equiv \Sigma_{i=1}^{k} n_i$. Then for the hypothesis $H(\underline{\xi}_1 = \cdots = \underline{\xi}_k) \equiv H_0$, the critical region (of size α) proposed was: $\theta_q \geqq \theta_0(\alpha, p, k-1, n)$, where $P[\theta_q \geqq \theta_0(\alpha, p, k-1, n) \mid H_0] = \alpha$, $q \equiv \min(p, k-1)$ and θ_q is the largest root of the determinantal equation: $\mid S^* - \theta S \mid = 0$. In this paper, denoting by $\underline{x}_i$ the p-dimensional mean column vector for the ith sample and inverting, in the usual manner, the process of construction of the above type of test, the following set of simultaneous confidence intervals (with a joint confidence coefficient $1 - \alpha$) is obtained: $\Sigma_{i=1}^{k} \lambda_i(n_i + 1)^{1/2} \underline{\mu}'(\underline{x}_i - \underline{\xi}_i) \leqq [k\theta_0(\alpha, p, k, n)\underline{\mu}'S\underline{\mu}]^{1/2}$ for all nonsingular column vectors $\underline{\mu}$ and all sets of $\lambda_i(i = 1, 2, \cdots, k)$ subject to: $\Sigma_i^k \lambda_i^2 = 1$. On the left hand side of the confidence interval statement the absolute value of a scalar is taken and on the right-hand side the positive square root of a positive quantity.

28. On a Relevant Set of Simultaneous Confidence Interval Statements in Discriminant Analysis. S. N. Roy and R. C. Bose, University of North Carolina.

In the setup of the previous abstract suppose that we are interested in a set of simultaneous confidence interval statements on $\underline{\mu}'(\underline{\xi}_i - \underline{\xi}_j)$ (for all nonsingular p-dimensional column vectors $\underline{\mu}$ and all $i \neq j = 1, 2, \cdots, k$). For a given pair (i, j) it is noted that

$$\frac{(n_1+1)(n_2+1)}{n_1+n_2+2} \max_{\underline{\mu}} \underline{\mu}'[\underline{x}_i - \underline{x}_j - (\underline{\xi}_i - \underline{\xi}_j)][\underline{x}_i' - \underline{x}_j' - (\underline{\xi}_i' - \underline{\xi}_j')]\underline{\mu} \div \underline{\mu}'S\underline{\mu}$$

$$= \frac{(n_1+1)(n_2+1)}{n_1+n_2+2} trS^{-1}(\underline{x}_i - \underline{x}_j - \underline{\xi}_i + \underline{\xi}_j)(\underline{x}_i' - \underline{x}_j' - \underline{\xi}_i' + \underline{\xi}_j') \equiv T_{ij};$$

and that this is distributed as Hotelling's T with D.F. p and $n + 1 - p$, where $n \equiv \Sigma_{i=1}^{k} n_i$. This is used to obtain finally the following set of simultaneous confidence intervals (for all $\underline{\mu}$ and $i \neq j = 1, 2, \cdots, k$) with a joint confidence coefficient $1 - \alpha$: $\underline{\mu}'(\underline{x}_i - \underline{x}_j - \underline{\xi}_i + \underline{\xi}_j) \leqq [\theta_0 \underline{\mu}'S\underline{\mu}]^{1/2}$, where θ_0 is given by the simultaneous probability:

$$P[T_{ij}(i \neq j = 1, \cdots, k) \leqq \theta_0] = 1 - \alpha.$$

While each T_{ij}, separately, is distributed as a Hotelling's T with D.F. p and $n + 1 - p$, the T_{ij}'s are not distributed independently and, before we could use the above, the distribution of $\max_{ij} T_{ij}$ has to be worked out. This is unlike the previous case where, on the null hypothesis, the distribution of θ_q is known and for which the construction of the necessary tables is now under way. It may be noted that the bunch of confidence intervals offered here is a subset of what is offered in the previous abstract.

29. The Separation of Product Error and Inspection Error in Sampling by Attributes. C. C. Craig, University of Michigan.

Though the separation of instrumental variability from that of the product being measured in case the measurements are values of a statistical variable of the continuous type has been dealt with, the analogous problem in case the inspection results consist only of the numbers of articles found acceptable in samples appears to have been neglected. The present paper gives the results of a straightforward approach to the problem including among the cases considered the realistic one in which the inspector rejects good items and accepts bad ones with unequal probabilities for the two kinds of error. Moment estimates which are also maximum likelihood estimates for these probabilities and for the process average fraction defective together with their large sample variances are found. However this appears to be an instance in which very large samples are required to give the asymptotic results practical value.

30. The Relation Between Fisher's Discriminant Function and Wald's Classification Statistic. H. LEON HARTER, Wright-Patterson Air Force Base.

A problem frequently encountered in statistics is that of classifying an individual into one of two populations, Π_1 or Π_2. Ordinarily, complete information about the populations is not available, but scores on p tests are known for samples of N_1 individuals from Π_1 and N_2 individuals from Π_2, as well as for the individual under consideration, a member of population Π. It is assumed known that Π is identical with either Π_1 or Π_2. It is required to test the hypothesis $H_1 : \Pi = \Pi_1$ against the single alternative $H_2 : \Pi = \Pi_2$. Fisher proposed a discriminant function X and Wald a classification statistic V for use in solving problems of this sort. In this paper it is shown that the relation between X and V takes the form $V = \alpha X - \beta$, where α depends on the sample sizes N_1 and N_2 and β on the hypothesis under consideration.

31. On the Distribution of the Ratio of the ith Observation in an Ordered Sample from a Normal Population to an Independent Estimate of the Standard Deviation. K. C. S. PILLAI AND K. V. RAMACHANDRAN, University of North Carolina.

Let $x_1 \cdots x_n$ be a sample of observations taken from a normal population arranged in the ascending order of magnitude and s, an independent estimate of the standard deviation with v degrees of freedom. The distribution of x_i/s is obtained as a series whose terms are Beta functions. The series is observed to be useful in small samples and with the help of the tables of incomplete Beta functions, the 5 per cent and 1 per cent levels of significance for x_n/s are being computed for small values of n and values of v up to 20.

32. Cyclic Solutions of Symmetrical Group Divisible Designs. S. S. SHRIKHANDE, University of Kansas.

An incomplete block design with v treatments each replicated r times in b blocks of size k is said to be group divisible (Bose and Connor, *Ann. Math. Stat.*, Vol. 23 (1952), pp. 367–383) if the treatments can be divided into m groups of n each so that any two treatments belonging to the same group occur together in λ_1 blocks, while any two treatments from different groups occur together in λ_2 blocks. The design is connected if $\lambda_2 > 0$. A set $(d_1, d_2, \cdots, d_r)$ of integers is called a cyclic difference set for the symmetrical group divisible design ($r = k$ and hence $b = v$) if block i is given by the set

$$(d_1 + i - 1, d_2 + i - 1, \cdots, d_r + i - 1)$$

reduced mod $v (i = 1, 2, \cdots, v)$. Two theorems on the impossibility of cyclic solutions are proved. Theorem 1. Let d be an odd positive factor of m, and f be a prime factor of the square-free part of $r^2 - \lambda_2 v$. Then there is no cyclic solution of the design if (i) $d = 3 \pmod 4$ and $(-d/f) = -1$ or (ii) $d = 1 \pmod 4$ and $(d/f) = -1$. Theorem 2. Let c be an odd positive factor of n when $(c, m) = 1$ and let ϕ be a prime factor of the square-free part of $r - \lambda_1$. Then there is no cyclic solution of the design if (i) $c = 3 \pmod 4$ and $(-c/\phi) = -1$ or (ii) $c = 1 \pmod 4$ and $(c/\phi) = -1$. These results are obtained by using a result due to Hall and Ryser (*Canadian Jour. Math.*, Vol. 3 (1951), pp. 495–502).

33. Measures of Association for Cross-Classifications. L. A. GOODMAN AND W. H. KRUSKAL, University of Chicago.

If a population is cross-classified by two classifications, one often desires a single number which describes the degree of association between the two classifications. Given such a measure of association based upon the population proportions, one may wish to estimate it or make tests about it on the basis of a sample drawn from the population in a specified way. Standard measures of association are described and criticized. A number of other measures are suggested and motivated in the frameworks of models for predictive behavior which seem typical of the uses to which cross-classifications are put. For example, one measure is based on the relative improvement in the prediction of one classification as the other is or is not known. Also discussed are measures of partial and multiple association if there are more than two classifications. The asymptotic sampling theory for certain measures and methods of sampling is discussed.

34. Calculating Longevities from Sample Composition. LEO A. GOODMAN, University of Chicago.

Sometimes it is desired to compare the longevity of two or more types of equipment under operational conditions where it is not convenient to identify or keep records of individual items. Such a comparison can be made by adopting certain replacement rules and observing their effect on the composition of the population. For example, when only two types are being compared, the replacement rule might be that when an item fails, its replacement will be of the opposite type. Then the composition of the population at any time (i.e., the proportions of the different types among all the items in use) will depend upon the original composition of the population, the time elapsed, and the longevities of the different types. Since the original composition and the elapsed time are known, by determining the new composition of the population (either by total inspection or by drawing a sample from the new composition) we would expect to obtain information concerning the longevities of the different types of equipment. Replacement policies are studied which satisfy certain logistics requirements as well as the requirement that a given number of items be in operation at all times. For certain given logistics requirements, optimum replacement rules are developed. The problem of estimating and testing hypotheses concerning the relative longevities of $K \geqq 2$ types of equipment is studied for the case where the equipment is subject to a constant risk. It is then shown that if the replacement rules are used for a long period of time, the results obtained under the assumption of constant risk remain valid even when the risk is not constant as long as the equipment has a finite life span. When information about the stock is also available (i.e., how many items have been replaced), estimates and tests of hypotheses concerning the longevities of the K types of equipment are obtained. A numerical illustration is given.

NEWS AND NOTICES

Readers are invited to submit to the Secretary of the Institute news items of interest

Personal Items

Dr. M. H. Belz has now returned to the University of Melbourne after spending the first semester at Princeton. During the period of his sabbatical leave he