The quadratic $\varphi = 1$ has solutions

$$\nu = \frac{3x \pm \sqrt{x^2 + 8}}{4}. \tag{10}$$

As ν is known to be greater than x, only the positive sign in (10) need be considered. The result so obtained is everywhere greater than x, and positive for all $x > -1$, giving the result

$$R_x < 4/\{3x + \sqrt{8 + x^2}\}, \qquad x > -1.$$

4. A corollary on the weight function in probit analysis. The function

$$\psi(x) = e^{-x^2} \Big/ \int_{-\infty}^{x} e^{-\frac{1}{2}u^2}\, du \int_{x}^{\infty} e^{-\frac{1}{2}u^2}\, du$$

is well known as the weight function in probit analysis. From tables it is obvious that ψ is a decreasing function of x^2. Hammersley [5] has given a rather complicated proof of this result, and has remarked on the apparent lack of a simple proof. In fact

$$\begin{aligned} \psi'(x) &= \psi(x)\{\nu(x) - \nu(-x) - 2x\} \\ &= 2x\psi(x)\{\lambda(x') - 1\}, \quad \text{where } -|x| \leqq x' \leqq |x|, \end{aligned}$$

by the Mean Value Theorem, and, since ψ is positive by definition, the result follows immediately from (3) above.

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ON A DOUBLE INEQUALITY OF THE NORMAL DISTRIBUTION[1]

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In this note we shall extend certain results of R. D. Gordon and Z. W. Birnbaum concerning bounds for the normal distribution function.

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Gordon [1] obtained the inequalities

$$\frac{x}{x^2+1}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} \leqq \int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\,dt \leqq \frac{1}{x}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} \quad \text{for} \quad x > 0.$$

Birnbaum [2] improved Gordon's lower bound, obtaining the inequality

$$\frac{\sqrt{4+x^2}-x}{2}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} \leqq \int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\,dt \quad \text{for} \quad x \geqq 0.$$

It was pointed out by Feller [3] that

$$\int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\,dt \sim \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\cdot\left\{\frac{1}{x}-\frac{1}{x^3}+\frac{1\cdot 3}{x^5}+\cdots+(-1)^k\frac{1\cdot 3\cdot\cdots(2k-1)}{x^{2k+1}}\right\},$$

where for $x > 0$ the right side is an upper bound when k is even and a lower bound when k is odd. It is evident that Feller's expression does not constitute an improvement of the bounds of Gordon and Birnbaum when $0 < x < 1$. The following theorem gives new bounds for

$$\int_x^\infty (1/\sqrt{2\pi})\, e^{-t^2/2}\,dt.$$

THEOREM:

$$\frac{1}{2}-\left(\frac{1}{4}-\frac{e^{-x^2}}{4}\right)^{\frac{1}{2}} \leqq \int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\,dt \leqq \frac{1}{2}+\frac{e^{-\frac{1}{2}x^2}}{x\sqrt{2\pi}}-\left(\frac{1}{4}+\frac{e^{-x^2}}{2\pi x^2}\right)^{\frac{1}{2}} \quad \text{for} \quad x \geqq 0$$

$$\frac{1}{2}+\frac{e^{-\frac{1}{2}x^2}}{x\sqrt{2\pi}}+\left(\frac{1}{4}+\frac{e^{-x^2}}{2\pi x^2}\right)^{\frac{1}{2}} \leqq \int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\,dt \leqq \frac{1}{2}+\left(\frac{1}{4}-\frac{e^{-x^2}}{4}\right)^{\frac{1}{2}} \quad \text{for} \quad x \leqq 0.$$

For the case $x \geqq 0$ the lower bound exceeds that of Birnbaum for some x and is exceeded by it for other values of x. The upper bound is an improvement on the result of Birnbaum and Gordon for all x. The inequalities for $x \leqq 0$ are of course obtainable immediately from the relation

$$\int_{-x}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\,dt = 1 - \int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\,dt.$$

The proof of the theorem will consist in proving two lemmas and then combining the results. In what follows we shall use the notation

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} \quad \text{and} \quad F(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\,dt.$$

LEMMA 1. $(2F-1)f \geqq F(1-F)x$ *for* $0 \leqq x < \infty$ *with equality at* 0 *and* ∞.

PROOF. Let $g = (2F-1)f - F(1-F)x$. Then,

$$g' = 2f^2 - F(1-F), \qquad g'' = -4xf^2 + (2F-1)f. \tag{1}$$

It may easily be shown that g is continuous with derivatives of all order, $g(0) = g(\infty) = 0$, and $g'(0) > 0$. From this we see that unless g is nonnegative for all positive x, there exists a minimum x_0 for which $g(x_0) < 0$. Now, from (1) and the definition of g, we have $g'' < F(1 - F)x_0 - 4x_0f^2 < -2x_0f^2 < 0$, which is impossible. Hence, g is nonnegative for all positive x, which completes the proof.

LEMMA 2. *$F(1 - F) \geqq \pi f^2/2$ for $0 \leqq x < \infty$ with equality at 0 and ∞.*

PROOF. Let $h = F(1 - F) - \pi f^2/2$. Then,

$$h' = f(1 - 2F) + \pi x f^2, \qquad h'' = f^2(\pi - 2 - 2\pi x^2) - xf(1 - 2F). \tag{2}$$

It may be shown that h is continuous with derivatives of all order, $h(0) = h(\infty) = 0$, $h'(0) = 0$, and $h''(0) > 0$. Let y_0 be an extremum of h. Then, from (2) $h'' = f^2(\pi - 2 - \pi y_0^2)$ at the point y_0. Hence, $y_0 \leqq (\pi - 2)^{\frac{1}{2}}/\sqrt{2}$ if y_0 is a minimum and $y_0 \geqq (\pi - 2)^{\frac{1}{2}}/\sqrt{2}$ is y_0 is a maximum, so that if a minimum and a maximum both exist, the minimum must precede the maximum. In view of this circumstance it is evident from the above mentioned properties of h, h' and h'' that a minimum cannot exist, and therefore that h is nonnegative for all positive x.

The results of Lemmas 1 and 2 can be rewritten respectively as

$$\left(F + \frac{f}{x} - \frac{1}{2}\right)^2 \geqq \left(\frac{f}{x} - \frac{1}{2}\right)^2 + \frac{f}{x}, \tag{3}$$

$$\left(F - \frac{1}{2}\right)^2 \leqq \frac{1}{4} - \frac{\pi}{2}f^2. \tag{4}$$

For $x \geqq 0$ the upper bound of the theorem is obtainable from (3) and the lower bound from (4).

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CORRECTION TO "SOME NONPARAMETRIC TESTS OF WHETHER THE LARGEST OBSERVATIONS OF A SET ARE TOO LARGE OR TOO SMALL"*

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This note calls attention to the fact that Theorem 4 of this paper (*Annals of Math. Stat.*, Vol. 21 (1950), pp. 583–592) is only partially correct. The results $\lim_{\Phi \to -\infty} P_1(\Phi) = 0$ and $\lim_{\Phi \to -\infty} P_3(\Phi) = 1$ as well as the monotonicity properties

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