NOTES

A NOTE ON INCOMPLETE BLOCK DESIGNS WITH ROW BALANCE*

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1. Introduction and summary. With the balanced incomplete block designs [1] r replicates of each of v treatments are arranged in b 'blocks', and the number of 'plots', k, in each block is smaller than v. In order to eliminate systematic block differences from the comparison of treatment means and to obtain treatment comparisons of equal precision the well known conditions of balance specify

(a) that no treatment should occur more than once in any block,

(b) that the number of blocks in which any two particular treatments are *both* applied should be a constant number of blocks (λ blocks) for all possible treatment pairs.

When these designs are used in this form, the position of treatments within each block is not specified and can normally be regarded as unimportant. Accordingly the treatments in each block are randomised. Situations, however, arise in which the 'plots' occupy certain characteristic positions in each block. Thus, if in a field experiment the blocks are vertical columns the plots would fall into k horizontal rows which may also have systematic effects on the yields. In this case it will often be advantageous to balance the design with regard to rows (as well as with regard to columns) in a manner similar to the Latin square. Such an arrangement was first developed by Youden [2] who used the particular incomplete block designs with $b = v$ and in these rearranged the treatments in each block in such a way that every treatment occurred precisely once in each row. More recently one of us (S. S. S.) has carried out similar rearrangements for the other incomplete block designs with $b > v$, $v \leqq 10$, $r \leqq 10$ (i.e., for those tabulated in standard tables and books), and has used a definition of balance resulting in at most two different precisions for treatment comparisons. In this note we show that

(i) balancing with regard to rows resulting in an *equal* precision of all treatment comparisons is possible if[1] $b = mv$ (m integral),

(ii) in all incomplete block designs with $r = mk \pm 1$ a row balance is possible resulting in treatment comparisons of *two* different precisions.

One of us (W. B. T.) has prepared complete tables of double balanced designs suitable for practical use which it is hoped to publish together with the analysis of variance procedure with recovery of interblock information.

2. Notations and preliminaries. We start from a balanced incomplete block design with parameters v, b, r, k and λ. It follows from (a) and (b) that each treat-

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[1] A theorem bearing on the necessity of this condition is given in Section 4.

ment occurs in just r blocks and that the integers v, b, r, k and λ must satisfy the following conditions.

$$vr = bk, \qquad \lambda(v - 1) = r(k - 1), \qquad b \geqq v. \tag{2.1}$$

The last inequality, which is not so obvious, is due to Fisher [4]. Let the design be written in row column form where the columns correspond to blocks and the rows to the positions of treatments in the blocks. Let n_{ij} be the number of times treatment i occurs in row j in the b columns ($i = 1, 2, \cdots v; j = 1, 2, \cdots k$). It is shown in [3] that if

$$\sum_j n_{ij}^2 = v, \qquad i = 1, 2, \cdots, v, \tag{2.2}$$

$$\sum_j n_{ij}\, n_{uj} = \mu, \qquad i \neq u = 1, 2, \cdots, v \tag{2.3}$$

then all treatment comparisons are made with equal accuracy. If, however, the design satisfies (2.2) and the condition

$$\sum_j n_{ij} n_{uj} = \mu_e, \qquad e = 1, 2, \tag{2.4}$$

where treatments i and u are e-associates as defined by Bose and Nair [5], then it has been shown there that there are two accuracies, that is, some pairs of treatments are compared with one accuracy while other pairs are compared with a different accuracy. The row balancing of all the designs obtained in the above paper [3] satisfies the conditions of partially balanced incomplete block designs as defined by Bose and Nair as well as that of a particular class of group divisible designs given by Nair and Rao [6]. In this particular class of incomplete designs the v treatments can be divided into c groups of size d each so that (2.2) holds and further

$$\sum_j n_{ij} n_{uj} = \mu_1 \text{ or } \mu_2 \tag{2.5}$$

according as treatments i and u belong to the same group or to different groups. It is easy to verify that such a group divisible design is always a partially balanced incomplete block design.

In the rest of the paper it is proved that if m is a positive integer then a balanced incomplete block design with $b = mv$ can be converted into a design satisfying (2.2) and (2.3) while a design with $r = mk \pm 1$ can be converted into a design with (2.2) and (2.5). These results follow from a general lemma due to Smith and Hartley [7] and which may be stated as follows:

LEMMA. *If we are given any set of bk elements made up of b "treatments" each replicated k times, the set being arranged in a two-way classification of k rows and b columns, then it is always possible to rearrange the elements in each column so that each row will contain every treatment once and only once.*

3. The case $b = mv$. Consider a balanced incomplete block design with parameters v, b, r, k and λ, where $b = mv$ and hence $r = mk$. Let it be written in row column form with k rows and b columns. Since $r = mk$ we can split the mk replications of each treatment into k replications of m pseudo-treatments. We then have a two-way classification of k rows and b columns in which each of the b $(= mv)$ pseudo-treatments are replicated k times. Applying the lemma above each of the b pseudo-treatments can be made to occur once and only once in every row. Hence each of the original v treatments can be made to occur m times in every row. We then get $\sum_j n_{ij}^2 = \sum_j n_{ij} n_{uj} = km^2$ for all i and u. Thus (2.2) and (2.3) are satisfied and, therefore, the converted design can be used for two-way elimination with the same accuracy for all treatment comparisons.

4. The case $mv < b < (m + 1)v$. In this case it is obviously impossible to specify that each treatment should occur the same number of times in every row since this would imply that r is a multiple of k and hence b is the same multiple of v. All one can hope for is that one may rearrange the treatments so that

CONDITION A. *Each treatment occurs either m or $m + 1$ times in every row, that is, $n_{ij} = m$ or $m + 1$ for all i and j.* Condition A implies that after rearrangement each treatment will occur $m + 1$ times in some $r - km$ rows and m times in the remaining ones. Hence condition (2.2) is satisfied with $\nu = 2rm + r - km^2 - km$.

One may inquire whether it is possible for a balanced incomplete block design with $mv < b < (m + 1)v$ to satisfy not only A but also condition (2.3), that is, $\sum_j n_{ij} n_{uj} = \mu$ for all i and u. If the design could be made to satisfy both these conditions then it could be used for two-way elimination of heterogeneity with the same accuracy for all treatment comparisons.

We now show that both these conditions cannot be simultaneously satisfied.

Suppose there actually exists a balanced incomplete block design with $mv < b < (m + 1)v$ satisfying both the conditions. Because of A there are in each row precisely $k' = b - mv$ treatments occurring $m + 1$ times and $v - k'$ treatments occurring m times in that row. Take one replicate from each of the former only, leaving out the remaining mv replicates of the v treatments. These form k "reduced" rows each of size k' in which each of the v treatments occur r' $(= r - mk)$ times. From $mv < b < (m + 1)v$ it follows that $r \geqq km + 1$ or $r' \geqq 1$. Further $vr' = kk'$ and $v > k$ imply that $k' > r'$ so that $k' \geqq 2$. The condition (2.3) then implies that for these reduced rows

$$\sum_{j=1}^{k'} n_{ij} n_{uj} = \mu' = \mu - km^2 - 2m(r - km), \qquad i \neq u = 1, 2, \cdots v,$$

where μ' necessarily integral is not less than 1. Since for these reduced rows $n_{ij} = 0$ or 1, it follows that every pair of treatments i and j will occur exactly in μ' of these reduced rows. Hence the array of k reduced rows would specify a balanced incomplete block design with parameters $v'(= v)$, $b'(= k)$, r', k' and μ'. But this is impossible from (2.1) since $b' < v'$.

We will now show that it is possible to convert balanced incomplete block designs with $r = mk \pm 1$ so that after suitable interchanges of treatments in various columns the v treatments are divided into c groups of d treatments each satisfying condition (2.2) and (2.5).

5. The case $r = mk + 1$. Since $r = mk + 1$ and $bk = vr$ it follows that $b = mv + t$ where $t = v/k$ is an integer. We now split the r replications of each treatment into m pseudo-treatments with k replications leaving one "odd" replication. The v "odd" replications can be considered as t further pseudo-treatments with k replications each. We now apply the lemma to this arrangement of b pseudo-treatments so that each pseudo-treatment occurs just once in every row. This implies that the original v treatments are divided into $c = k$ groups of $d = t$ each so that in each row all the treatments of only one group occur $m + 1$ times each, while the remaining ones occur m times. Further, treatments of any group occur $m + 1$ times each in only one row. It is easily verified that (2.3) and (2.5) are satisfied with $\nu = \mu_1 = m^2k + 2m + 1$ and $\mu_2 = m^2k + 2m$.

6. The case $r = mk - 1, m \geqq 2$. In this case $b = mv - t$ where $t = v/k$ is an integer. We now split the r replication of each treatment into $m - 1$ $(\geqq 1)$ pseudo-treatments with k replications each leaving $k - 1$ "odd" replications. Now add t dummy blocks of size k containing each of the v treatments precisely once. The v treatments may be arranged in any way in these dummy blocks. The $k - 1$ "odd" replications of any treatment together with the replication of the same treatment in the dummy blocks can be considered as one more pseudo-treatment with k replications. Thus in all we have mv pseudo-treatments occurring k times in the mv blocks. We now apply the lemma to these mv blocks. After the lemma's rearrangement we consider only the b original blocks, and find that each of the v original treatments occurs either $m - 1$ or m times in every row. We now divide the v treatments into k groups of size t by placing those treatments, which occur $m - 1$ times in a particular row, in one group. It is easily seen that (2.2) and (2.3) are then satisfied with $\nu = \mu_1 = km^2 - 2m + 1$ and $\mu_2 = km^2 - 2m$.

Finally it should be noted that the interchanges discussed in Sections 3–6 do not require that every pair of treatments should occur in the same number of columns. It is sufficient that every treatment occurs the same number of times in the b columns.

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