SOME TESTS BASED ON ORDERED OBSERVATIONS FROM TWO EXPONENTIAL POPULATIONS[1]

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1. Introduction. Let $x_{11} \leqq x_{12} \leqq \cdots \leqq x_{1n_1}$, and $x_{21} \leqq x_{22} \leqq \cdots \leqq x_{2n_2}$, be two random samples (S_{n_1} and S_{n_2}) from populations having p.d.f.'s $f(x; A_1, \theta_1)$ and $f(x; A_2, \theta_2)$ respectively, where

$$f(x; A, \theta) = \frac{1}{\theta} \exp [-(x - A)/\theta]. \tag{1}$$

Let S_{r_1} and S_{r_2} be the sets of the first r_1 and r_2 smallest observations of S_{n_1} and S_{n_2} respectively. Then the p.d.f.'s of S_{r_1} and S_{r_2} are given, say, by

$$g(x_{11}, \cdots, x_{1r_1}; A_1, \theta_1) \quad \text{and} \quad g(x_{21}, \cdots, x_{2r_2}; A_2, \theta_2),$$

where

$$\begin{aligned} &g(x_1, x_2, \cdots, x_r; A, \theta) \\ &= \frac{n!}{(n-r)!} \frac{1}{\theta^r} \exp\left\{-\frac{1}{\theta}\left[\sum_{i=1}^{r} (x_i - A) + (n - r)(x_r - A)\right]\right\}. \end{aligned} \tag{2}$$

The likelihood ratio tests based on the complete sets, S_{n_1} and S_{n_2} are special cases of those obtained by Sukhatme [2], [3]. It can be shown that similar likelihood ratio tests based on S_{r_1} and S_{r_2} may be obtained by following Sukhatme's procedure [2]. In this paper these likelihood ratio tests are reduced to equivalent tests which are expressed in terms of the well known chi square and Snedecor's F distributions. Furthermore, some of the tests obtained in this paper can be extended to k-sample tests.

Since percentage points for χ^2 and F distributions are tabled, tests involving these random variables are useful in applications. We remark that the likelihood ratio test for the hypothesis H_5 (see Section 3) has been obtained by Paulson [1].

The results of this paper can be used in the field of life testing. A characteristic feature of such tests is that observations become available in order of magnitude. The assumption of an exponential distribution of life is a reasonable one to make in some applications (e.g., electron tube life). The parameter A can be interpreted as minimum life (also called sensitivity limit in fatigue failure problems) and the parameter θ is the mean life measured from A as a starting point. From the life test point of view one has a sample of size n_1 from population 1 and a sample of size n_2 from population 2, the two populations one wishes to compare. Procedures

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are given for testing various hypotheses regarding the A_i and θ_i $(i = 1, 2)$ based on information which has been truncated in the sense that one has only the first r_1 failure times from the sample of size n_1 (population 1) and the first r_2 failure times from the sample of size n_2 (population 2). r_1 and r_2 (as well as n_1 and n_2) are assumed to be preassigned.

2. Preliminary lemmas. We give several lemmas which were used to obtain the distributions of the reduced statistics. Lemmas 1 and 2 can be proved by the use of characteristic functions and their proofs are omitted. Proofs of Lemmas 3 to 6 are given.

In Lemmas 1 and 2 below, we let $x_1 \leqq x_2 \leqq \cdots \leqq x_r \leqq \cdots \leqq x_n$ be a random sample from a population having p.d.f. (1) and we define statistics u, v, and h as,

$$u = \frac{2}{\theta}\left[\sum_{i=1}^{r} (x_i - A) + (n - r)(x_r - A)\right]. \tag{3}$$

$$v = \frac{2}{\theta}\left[\sum_{i=1}^{r} (x_i - x_1) + (n - r)(x_r - x_1)\right]. \tag{4}$$

$$h = \frac{2n}{\theta}(x_1 - A). \tag{5}$$

LEMMA 1. *u is distributed as $\chi^2(2r)$.*

LEMMA 2. *v and h are independently distributed as $\chi^2(2r - 2)$ and $\chi^2(2)$ respectively.*

Lemmas 3 to 6 deal with the case of two samples. The statistics u_1, v_1 and u_2, v_2 are defined as in (3) and (4). Three additional variables w_1, w_2, and w are defined in (6), (7), and (8).

$$w_1 = \frac{2n_1}{\theta_1}(x_{11} - x_{21}), \qquad \text{for} \qquad x_{11} > x_{21} \tag{6}$$

$$w_2 = \frac{2n_2}{\theta_2}(x_{21} - x_{11}), \qquad \text{for} \qquad x_{21} > x_{11} \tag{7}$$

$$w = w_1, \quad \text{when} \quad x_{11} > x_{21} \qquad \text{and} \qquad w = w_2, \quad \text{when} \quad x_{21} > x_{11}. \tag{8}$$

LEMMA 3. *If $A_1 = A_2$, then*

$$\Pr(x_{11} > x_{21}) = \frac{n_2/\theta_2}{n_1/\theta_1 + n_2/\theta_2} \tag{9}$$

and

$$\Pr(x_{21} > x_{11}) = \frac{n_1/\theta_1}{n_1/\theta_1 + n_2/\theta_2}. \tag{10}$$

PROOF.

$$\Pr(x_{11} > x_{21}) = \int_{A_1}^{\infty} \int_{A_1}^{x_{11}} \frac{n_1 n_2}{\theta_1 \theta_2} e^{-(n_1/\theta_1)(x_{11}-A_1)-(n_2/\theta_2)(x_{21}-A_1)} \, dx_{11} \, dx_{21}$$

$$= \frac{n_2/\theta_2}{n_1/\theta_1 + n_2/\theta_2}.$$

Hence,

$$\Pr(x_{21} > x_{11}) = 1 - \Pr(x_{11} > x_{21}) = \frac{n_1/\theta_1}{n_1/\theta_1 + n_2/\theta_2}.$$

LEMMA 4. *If* $A_1 = A_2$, *then both* w_1 *(given that* $x_{11} > x_{21}$*) and* w_2 *(given that* $x_{21} > x_{11}$*) are distributed as* $\chi^2(2)$.

PROOF. Since $A_1 = A_2$, w_1 can be written as

$$w_1 = \frac{2n_1}{\theta_1} [(x_{11} - A_1) - (x_{21} - A_2)].$$

Consequently,

$$x_{11} - A_1 = \frac{\theta_1}{2n_1} w_1 + (x_{21} - A_2).$$

Let $x_{11} - A_1 = y_1$ and $x_{21} - A_2 = y_2$, then the condition that $x_{11} > x_{21}$ is equivalent to $y_1 > y_2$. Since the joint distribution of y_1 and y_2 is, say

$$f(y_1, y_2) = \frac{n_1 \cdot n_2}{\theta_1 \cdot \theta_2} e^{-(n_1/\theta_1)y_1-(n_2/\theta_2)y_2}, \quad y_1, y_2 > 0, \tag{11}$$

we have

$$\Pr(w_1 \leqq w_{10}, y_1 > y_2) = \frac{n_2/\theta_2}{n_1/\theta_1 + n_2/\theta_2} [1 - e^{-w_{10}/2}]. \tag{12}$$

According to Lemma 3

$$\Pr(y_1 > y_2) = \Pr(x_{11} > x_{21}) = \frac{n_2/\theta_2}{n_1/\theta_1 + n_2/\theta_2}. \tag{13}$$

Therefore,

$$\Pr(w_1 \leqq w_{10} \mid y_1 > y_2) = 1 - e^{-w_{10}/2} \tag{14}$$

which is the cumulative form of the χ^2 distribution with 2 d.f. This completes the proof of the first assertion in Lemma 4. The proof for the second assertion is similar.

LEMMA 5. *If* $A_1 = A_2$, *then* w *is distributed as* $\chi^2(2)$.

PROOF. Since

$$\Pr(w \leqq w_0) = \Pr(w_1 \leqq w_0, y_1 > y_2) + \Pr(w_2 \leqq w_0, y_1 < y_2) \tag{15}$$

then by (12)

$$\Pr(w \leqq w_0) = 1 - e^{-w_0/2} \tag{16}$$

which proves Lemma 5.

LEMMA 6. *If* $A_1 = A_2$, *then* (a) v_1, v_2, *and* w_1 *(given that* $x_{11} > x_{21}$*), or* (b) v_1, v_2, *and* w_2 *(given that* $x_{11} < x_{21}$*), or* (c) v_1, v_2, *and* w *are independently distributed as* $\chi^2(2r_1 - 2)$, $\chi^2(2r_2 - 2)$ *and* $\chi^2(2)$ *respectively.*

PROOF. By Lemma 2, v_1 and v_2 are each independent of both x_{11} and x_{21} and the results follow using Lemma 5.

3. Likelihood ratio tests and equivalent reduced tests. The various hypotheses and their associated likelihood ratio and equivalent reduced tests are listed below in Sections A, B, and C. One of the derivations will be given in Section D. Some properties of the tests are given in Section E.

A. Statement of hypotheses.

a) H_1 : To test $\theta_1 = \theta_2$
(assuming A_1 and A_2 are known).

b) H_2 : To test $\theta_1 = \theta_2$
(assuming $A_1 = A_2$, but that the common value is unknown).

c) H_3 : To test $\theta_1 = \theta_2$.

d) H_4 : To test $A_1 = A_2$
(assuming θ_1 and θ_2 are known).

e) H_5 : To test $A_1 = A_2$
(assuming $\theta_1 = \theta_2$, but that the common value is unknown).

f) H_6 : To test $A_1 = A_2$.

g) H_7 : To test $A_1 = A_2$ and $\theta_1 = \theta_2$.

B. Likelihood ratio tests.

In a), b) and c) below we let

$$K = \prod_{i=1}^{2} \left(\frac{r_1 + r_2}{r_i}\right)^{r_i}. \tag{19}$$

a) For H_1 :

$$\lambda_1 = K\left[(1 + c_1)^{r_1}\left(1 + \frac{1}{c_1}\right)^{r_2}\right]^{-1} \tag{20}$$

where

$$c_1 = \left[\sum_{j=1}^{r_2} (x_{2j} - A_2) + (n_2 - r_2)(x_{2r_2} - A_2)\right] \Bigg/ \left[\sum_{j=1}^{r_1} (x_{1j} - A_1) + (n_1 - r_1)(x_{1r_1} - A_1)\right]. \tag{21}$$

b) For H_2:

$$\lambda_2 = K\left[(1 + c_2)^{r_1}\left(1 + \frac{1}{c_2}\right)^{r_2}\right]^{-1}, \quad \text{if} \quad x_{11} < x_{21} \tag{22}$$

$$= K\left[\left(1 + \frac{1}{c_2'}\right)^{r_1}(1 + c_2')^{r_2}\right]^{-1}, \quad \text{if} \quad x_{21} < x_{11}$$

where

$$c_2 = \left[\sum_{j=1}^{r_2}(x_{2j} - x_{11}) + (n_2 - r_2)(x_{2r_2} - x_{11})\right] \Big/ \left[\sum_{j=1}^{r_1}(x_{1j} - x_{11}) + (n_1 - r_1)(x_{1r_1} - x_{11})\right] \tag{23}$$

$$c_2' = \left[\sum_{j=1}^{r_1}(x_{1j} - x_{21}) + (n_1 - r_1)(x_{1r_1} - x_{21})\right] \Big/ \left[\sum_{j=1}^{r_2}(x_{2j} - x_{21}) + (n_2 - r_2)(x_{2r_2} - x_{21})\right].$$

c) For H_3:

$$\lambda_3 = K\left[(1 + c_3)^{r_1}\left(1 + \frac{1}{c_3}\right)^{r_2}\right]^{-1} \tag{24}$$

where

$$c_3 = \left[\sum_{j=1}^{r_2}(x_{2j} - x_{21}) + (n_2 - r_2)(x_{2r_2} - x_{21})\right] \Big/ \left[\sum_{j=1}^{r_1}(x_{1j} - x_{11}) + (n_1 - r_1)(x_{1r_1} - x_{11})\right]. \tag{25}$$

d) For H_4:

$$\lambda_4 = e^{-c_4/2} \tag{26}$$

where

$$c_4 = w. \tag{27}$$

e) For H_5:

$$\lambda_5 = (1 + c_5)^{-(r_1+r_2)}, \quad \text{if } x_{11} > x_{21} \tag{28}$$

$$= (1 + c_5')^{-(r_1+r_2)}, \quad \text{if } x_{11} < x_{21}$$

where

$$c_5 = [n_1(x_{11} - x_{21})] \Big/ \left(\sum_{i=1}^{2}\left[\sum_{j=1}^{r_i}(x_{ij} - x_{i1}) + (n_i - r_i)(x_{ir_i} - x_{i1})\right]\right) \tag{29}$$

$$c_5' = [n_2(x_{21} - x_{11})] \Big/ \left(\sum_{i=1}^{2}\left[\sum_{j=1}^{r_i}(x_{ij} - x_{i1}) + (n_i - r_i)(x_{ir_i} - x_{i1})\right]\right)$$

f) For H_6:

$$\text{(30)}\qquad \begin{aligned} \lambda_6 &= (1 + c_6)^{-r_1}, && \text{if } x_{11} > x_{21} \\ &= (1 + c_6')^{-r_2}, && \text{if } x_{11} < x_{21} \end{aligned}$$

where

$$\text{(31)}\qquad \begin{aligned} c_6 &= [n_1(x_{11} - x_{21})] \Big/ \left[\sum_{j=1}^{r_1} (x_{1j} - x_{11}) + (n_1 - r_1)(x_{1r_1} - x_{11}) \right] \\ c_6' &= [n_2(x_{21} - x_{11})] \Big/ \left[\sum_{j=1}^{r_2} (x_{2j} - x_{21}) + (n_2 - r_2)(x_{2r_2} - x_{21}) \right]. \end{aligned}$$

g) For H_7:

$$\text{(32)}\qquad \lambda_7 = \prod_{i=1}^{2} \left(\frac{\hat{\theta}_i}{\hat{\theta}} \right)^{r_i}$$

where

$$\text{(33)}\qquad \begin{aligned} \hat{\theta}_i &= \frac{1}{r_i} \left[\sum_{j=1}^{r_i} (x_{ij} - x_{i1}) + (n_i - r_i)(x_{ir_i} - x_{i1}) \right] \\ \hat{\theta} &= \frac{1}{r_1 + r_2} \sum_{i=1}^{2} \left[\sum_{j=1}^{r_i} (x_{ij} - \hat{A}) + (n_i - r_i)(x_{ir_i} - \hat{A}) \right] \end{aligned}$$

and where $\hat{A} = \min (x_{11}, x_{21})$.

C. Reduced Tests.

By the use of the lemmas in Section 2, $\lambda_1, \lambda_2, \cdots, \lambda_6$ can be reduced to the following equivalent tests having the corresponding distributions (see Table 1). The authors have not succeeded in reducing λ_7 to an F-test or a χ^2-test (as was possible in $\lambda_1, \lambda_2, \cdots, \lambda_6$). We should like to mention, however, that in [3] Sukhatme found a cdf for λ_7 in the special case where $r_1 = n_1$ and $r_2 = n_2$. If further $n_1 = n_2$ the cdf he obtained involves the inverse hyperbolic cosine. Undoubtedly one can obtain a similar result for the cdf of λ_7 especially if $r_1 = r_2$.

In Table 1, numbers in the "critical regions" column indicate that the reduced tests obtained may be either one-sided or two-sided. For example, consider the case where $r_1 = r_2 = 10$ and $\alpha = .05$. Then for the various H_i, $i = 1, 2, 3, 4, 5, 6$, we have the following critical regions which are summarized for convenience in Table 2.

It should be noted that under H_2, H_4, H_5, and H_6 the distribution of the appropriate λ criteria consists of two parts, that is, depending on whether $x_{11} > x_{21}$ or $x_{11} < x_{21}$. In order to maintain a λ criterion in the form $0 < \lambda < c$, for some appropriate constant c, we should use (if $r_1 = r_2$) critical regions of size α in each of the two parts. If $r_1 \neq r_2$, then, in the case of H_6 one has to abandon the λ criterion in the simple form just given, because $\Pr(x_{11} > x_{21})$ is unknown. In order to obtain a test which is of size α, the statistician is forced to use critical regions of size α in each of the two parts. If $r_1 \neq r_2$ and one is dealing with H_2, then it is possible (but not easy) to maintain the λ criterion in its usual form. However it seems appropriate on practical grounds to use critical regions of equal size α in each of the two parts.

D. Derivation of the test under H_2.

TABLE 1

Hypothesis	Equivalent Reduced Tests	Distributions	Critical Regions
H_1	$f_1 = \frac{r_1}{r_2} c_1$	$F(2r_2, 2r_1)$	(2)
H_2	$f_2 = \frac{r_1 - 1}{r_2} c_2$, if $x_{11} < x_{21}$	$F(2r_2, 2r_1 - 2)$	(2)
	$f_2' = \frac{r_2 - 1}{r_1} c_2'$, if $x_{21} < x_{11}$	$F(2r_1, 2r_2 - 2)$	(2)
H_3	$f_3 = \frac{r_1 - 1}{r_2 - 1} c_3$	$F(2r_2 - 2, 2r_1 - 2)$	(2)
H_4	$f_4 = c_4$	$\chi^2(2)$	(1)
H_5	$f_5 = \frac{2r_1 + 2r_2 - 4}{2} c_5$, if $x_{11} > x_{21}$	$F(2, 2r_1 + 2r_2 - 4)$	(1)
	$f_5' = \frac{2r_1 + 2r_2 - 4}{2} c_5'$, if $x_{21} > x_{11}$	$F(2, 2r_1 + 2r_2 - 4)$	(1)
H_6	$f_6 = \frac{2r_1 - 2}{2} c_6$, if $x_{11} > x_{21}$	$F(2, 2r_1 - 2)$	(1)
	$f_6' = \frac{2r_2 - 2}{2} c_6'$, if $x_{21} > x_{11}$	$F(2, 2r_2 - 2)$	(1)

Since the derivations are similar in all cases, it would be sufficient to give one of them as an illustration. For the case H_2, the proof is as follows.

Assuming $A_1 = A_2 = A$, then the likelihood function is given by

$$h(x_{11}, \cdots, x_{1r_1}, x_{21}, \cdots, x_{2r_2}; A, \theta_1, \theta_2)$$

$$(34) \qquad = \prod_{i=1}^{2} \frac{n_i!}{(n_i - r_i)!} \cdot \frac{1}{\theta_i^{r_i}} \exp\left\{-\frac{1}{\theta_i}\left[\sum_{j=1}^{r_i} (x_{ij} - A) + (n_i - r_i)(x_{ir_i} - A)\right]\right\}.$$

TABLE 2
Critical Regions

H_1:	$f_1 > 2.46$ or $f_1 < \frac{1}{2.46}$
H_2:	$f_2 > 2.56$ or $f_2 < \frac{1}{2.50}$ when $x_{11} < x_{21}$ and $f_2' > 2.56$ or $f_2' < \frac{1}{2.50}$ when $x_{21} < x_{11}$
H_3:	$f_3 > 2.60$ or $f_3 < \frac{1}{2.60}$
H_4:	$f_4 > 5.99$
H_5:	$f_5 > 3.26$ when $x_{11} > x_{21}$ and $f_5' > 3.26$ when $x_{21} > x_{11}$
H_6:	$f_6 > 3.55$ when $x_{11} > x_{21}$ and $f_6' > 3.55$ when $x_{21} > x_{11}$

In the whole parameter space Ω: $A, \theta_1, \theta_2 > 0$ we obtain the maximum likelihood estimates

(35) $$\hat{A} = \min. (x_{11}, x_{21})$$

(36) $$\hat{\theta}_i = \frac{1}{r_i}\left[\sum_{j=1}^{r_i} (x_{ij} - \hat{A}) + (n_i - r_i)(x_{ir_i} - \hat{A})\right].$$

In the subspace ω: $A > 0$, $\theta_1 = \theta_2 = \theta > 0$ we have

(37) $$\hat{A} = \min (x_{11}, x_{21})$$

(38) $$\hat{\theta} = \frac{1}{r_1 + r_2}\sum_{i=1}^{2}\left[\sum_{j=1}^{r_i} (x_{ij} - \hat{A}) + (n_i - r_i)(x_{ir_i} - \hat{A})\right].$$

Hence, it can be easily verified that the likelihood ratio is given by λ_2 as in (22).

Further, λ_2 is a function of c_2 (or c_2'), which under H_2 can be written as

(39) $$c_2 = \frac{v_2 + w_2}{v_1}, \qquad \text{if } x_{11} < x_{21}$$

(40) $$c_2' = \frac{v_1 + w_1}{v_2}, \qquad \text{if } x_{21} < x_{11}.$$

Consequently, the reduced test given in Table I follows from Lemma 6 and other standard results on the distribution of the sum and ratio of independent chi squares.

E. Some properties of the various tests.

A number of properties for the various reduced tests are of interest. Some of them are:

(a) The tests (critical regions on λ) for H_1, H_3, H_4, H_5, do not depend on n_1 and n_2. This statement is also true for H_2 and H_6 if $r_1 = r_2$.

(b) The power of the tests for H_1 and H_3 and also for H_2 if $r_1 = r_2$ does not depend on n_1 or n_2.

(*c*) The tests are unbiased.

(d) The power of the tests for H_2 and H_3 is independent of A_1 and A_2.

These properties are fairly obvious. There are other properties which can be discovered by a more detailed investigation.

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