RANDOM FUNCTIONS SATISFYING CERTAIN LINEAR RELATIONS

By S. G. Ghurye

University of North Carolina[1]

Introduction and Summary. A hypothesis often made about a sequence of real-valued r.v. (random variables), $\{X_n, n = 0, \pm 1, \pm 2, \cdots\}$ is that there exist certain real constants $\alpha_1, \alpha_2, \cdots, \alpha_k$ such that if we write

$$Y_n = X_n + \alpha_1 X_{n-1} + \cdots + \alpha_k X_{n-k}, \tag{1}$$

then $\{Y_n\}$ is a sequence of independent r.v. Now, very often in practice, the observed sequence $\{X_n\}$ consists in observations made at equidistant t-points on a stochastic process with a continuous parameter t. Restricting our attention to the case $k = 1$, we then have a r.f. (random function) $X(t)$, defined for all t in an interval, with the following property: There exist a value of t (t_0, say), and $h > 0$, and a real number α such that a hypothesis of the type mentioned is satisfied by the sequence $\{Y(t_0, h; n), n = 0, \pm 1, \pm 2, \cdots\}$, where

$$Y(t_0, h; n) = X(t_0 + nh) - \alpha X(t_0 + [n - 1]h). \tag{2}$$

But if such a hypothesis is true for one value of h, it is not necessarily so for some other value; and we have to make the additional assumption that the particular length of the t-intervals with which we are concerned is precisely the one for which the hypothesis holds. This assumption may not be reasonable in every case, Instead we may wish to work with a hypothesis similar to the above, but which holds for all positive h in some interval.

In this paper, we investigate the existence and form of random functions satisfying a hypothesis of this type. Section 1 contains a statement of the problem and some simple results. It turns out that any random function possessing the required property can be expressed as the product of an exponential function of t and a r.f. with independent increments. Section 2 deals with the limit in distribution of a sequence of Stieltjes approximating sums involving a r.f. with independent increments. Finally, in Section 3, these results are applied to the problem under investigation, and further possibilities are investigated.

It must be noted that, in this investigation, we are concerned only with results in distribution. That is to say, we are dealing throughout only with parametric families of probability laws and talking in terms of r.f. merely for convenience.

1. The problem and some preliminary results. We consider a real-valued random vector function $X(t)$ whose transpose $X'(t)$ is the row vector $\{X^{(1)}(t), \cdots, X^{(p)}(t)\}$ defined and continuous in probability for all $t \geqq$ some t_0.

Received 7/24/53.

[1] Work done under the sponsorship of the Office of Naval Research.

We suppose that there exists a real-valued, $p \times p$ matric function $A(h) = \{\alpha^{ij}(h)\}$ defined and continuous for $h \geqq 0$, and such that if we write

$$Y(n; h) = X(t_0 + nh) - A(h)X[t_0 + (n - 1)h], \tag{3}$$

then for any $h > 0$ and any integer N, $X(t_0)$, $Y(1; h)$, $\cdots$, $Y(N; h)$ are mutually independent.

We shall be concerned with the existence of such random functions $X(t)$ and with the functional form of $A(h)$, the probability law of $X(t)$, and other questions of this sort. To start with, we shall prove

LEMMA 1. *Let $X' = \{X^{(1)}, \cdots, X^{(m)}\}$ and $Y' = \{Y^{(1)}, \cdots, Y^{(n)}\}$ be independent random vectors, and let there exist an $m \times n$ matrix A of rank r such that $X + AY$ is independent of Y. Then there exist at least r linearly independent n-vectors $c(j)$ such that, with probability 1, $c'(j)Y$ is a constant, $j = 1, 2, \cdots, r$.*

PROOF. Let

$$t' = \{t^{(1)}, \cdots, t^{(m)}\}, \qquad u' = \{u^{(1)}, \cdots, u^{(n)}\}, \qquad f(t, u) = E\{e^{i(t'X+u'Y)}\}.$$

Then the independence of X and Y implies

$$f(t, u) = f(t, 0)f(0, u), \qquad f(t, A't + u) = f(t, 0)f(0, A't + u).$$

From the independence of $X + AY$ and Y, we have

$$f(t, A't + u) = f(t, A't)f(0, u) = f(t, 0)f(0, A't)f(0, u).$$

Since $f(0, 0) = 1$ and $f(t, u)$ is continuous, there exists a region, $t't \leqq T^2$, in which $f(t, 0) \neq 0$. Consequently, for any t in this region and any u, we have

$$f(0, A't + u) = f(0, A't)f(0, u).$$

Since A' is $n \times m$ and of rank r, there exist nonsingular matrices B and C such that

$$A' = C' \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right) B'.$$

Let

$$B't = v, \qquad u = C'w, \qquad \bar{v}' = \{v^{(1)}, \cdots, v^{(r)}, 0, \cdots, 0\},$$
$$\bar{w}' = \{w^{(1)}, \cdots, w^{(r)}, 0, \cdots, 0\}, \qquad f(0, C'\bar{v}) = g(\bar{v}).$$

Then, for any $\bar{v}$ in a certain neighbourhood of the origin and for any $\bar{w}$,

$$g(\bar{v} + \bar{w}) = g(\bar{v})g(\bar{w}) \qquad \text{and} \qquad g(\bar{v}) = e^{(\alpha' + i\beta')\bar{v}}$$

Since g is a characteristic function, $\alpha = 0$ and $f(0, C'\bar{v}) = e^{i\beta'\bar{v}}$. It follows that if $c(j)$ be the jth column of C', then

$$\Pr\{c'(j)Y = \beta^{(j)}\} = 1, \qquad j = 1, 2, \cdots, r.$$

COROLLARY 1. *If Y is not linearly singular, in the sense that there is no $c'Y$ which is a constant with probability 1, if $X + AY$ is independent of Y, and $X + BY$ is independent of Y, then $A = B$.*

THEOREM 1. *Let $X(t)$ and $A(h)$ be as assumed at the start of the section, with* *the additional conditions that $X(t_0)$ is not linearly singular and $A(h)$ is nonsingular for some $h > 0$.*

Then $A(h)$ can be written in the form $e^{\Lambda h}$, where Λ is a constant matrix. Further $X^*(t) = e^{-\Lambda t}X(t)$ is an additive process. Conversely, if Λ is any constant matrix and $X^*(t)$ an additive process, $X(t) = e^{\Lambda t}X^*(t)$ is of the desired type with $A(h) = e^{\Lambda h}$.

PROOF. For any $h > 0$ and any m, $X(t_0)$, $Y(1; hm^{-1}), \cdots, Y(j; hm^{-1}), \cdots, Y(m; hm^{-1})$ are independent. Therefore $\sum_{j=0}^{m-1} \{A(hm^{-1})\}^j Y(m - j; hm^{-1})$ is independent of $X(t_0)$; that is, $X(t_0 + h) - \{A(hm^{-1})\}^m X(t_0)$ is independent of $X(t_0)$. Hence by Corollary 1, $\{A(hm^{-1})\}^m = A(h)$. It follows, by the usual continuity argument, that $A(h_1)A(h_2) = A(h_1 + h_2)$.

Let h_1 be such that $A(h_1)$ is nonsingular, and let $h_2 \rightarrow 0$. Then $\lim_{h\rightarrow 0} A(h) = A(0)$ is the identify matrix I. Nagy [11] has shown that, under these conditions. there exists a constant matrix Λ such that $A(h) = e^{\Lambda h}$ and

$$\Lambda = \lim_{h\rightarrow 0} \{A(h) - I\}h^{-1}.$$

Now let $t_0 < t_1 < t_1 + h_1 \leqq t_2 < t_2 + h_2$, and $t_1 - t_0 = h$. Then $X(t_0)$, $Y(j; hn^{-1})$, $j = 1, 2, \cdots, n, n + 1, \cdots$, are independent r.v. Hence

$$(4) \qquad X(t_0) \qquad \begin{matrix} X(t_1 + m_1hn^{-1}) - A(m_1hn^{-1})X(t_1), \\ X(t_1 + m_3hn^{-1}) - A\{(m_3 - m_2)hn^{-1}\}X(t_1 + m_2hn^{-1}), \end{matrix}$$

are mutually independent if $m_1 \leqq m_2 < m_3$. By letting n, m_1, m_2, $m_3 \rightarrow \infty$ in a suitable manner, we can show that the r.v. (4) converge in distribution respectively to

$$(5) \qquad X(t_0), \qquad X(t_1 + h_1) - A(h_1)X(t_1), \qquad X(t_2 + h_2) - A(h_2)X(t_2).$$

Hence, these are mutually independent. In the same manner, we see that for any integer k and any $t_1, t_2, \cdots, t_k$ in $[t_0, t]$, $X(t + h) - A(h)X(t)$ is independent of $X(t_1), \cdots, X(t_k)$, so that $X^*(t + h) - X^*(t)$ is independent of $X^*(t_1), \cdots, X^*(t_k)$. In other words, $X^*(t)$ is a random function with independent increments which are also independent of $X^*(t_0)$. For convenience, we shall use Paul Lévy's terminology and call $X^*(t)$ an additive process.

The final, converse statement of the theorem is obvious.

2. Concerning additive processes. The general form of the l.c.f. (logarithm of the characteristic function) of an additive process $Z(t)$ which is continuous in probability has been derived by Paul Lévy [7] and [8] (for alternate derivations and forms, see Doob [1], Feller [2], Gnedenko [4] and Khintchine [6]). It may be expressed as follows.

Let $u' = \{u^{(1)}, \cdots, u^{(p)}\}$ be a vector variable, and t_1, t_2 be any real numbers ($t_1 < t_2$). Then

$$(7) \qquad \begin{cases} \log E\{e^{iu'[Z(t_2)-Z(t_1)]}\} = \psi(u; t_2) - \psi(u; t_1), \\ \psi(u; t) = iu'\mu(t) - \tfrac{1}{2}u'\Sigma(t)u \\ \qquad + \displaystyle\int_{R(p)} \{e^{iu'x} - 1 - iu'x(1 + x'x)^{-1}\}(1 + x'x)(x'x)^{-1}\, d_xG(x; t), \end{cases}$$

when $R(p)$ is the whole p-dimensional Euclidean space of x. Also, both $\mu'(t) = \{\mu^{(1)}(t), \cdots, \mu^{(p)}(t)\}$ and

$$\Sigma(t) = \begin{pmatrix} \sigma^{11}(t) & \cdots & \sigma^{1p}(t) \\ \cdots & \cdots & \cdots \\ \sigma^{p1}(t) & \cdots & \sigma^{pp}(t) \end{pmatrix} = \Sigma'(t)$$

are continuous, and for every t and any $h > 0$, $\Sigma(t+h) - \Sigma(t)$ is nonnegative definite. Further, $G(x; t)$ is a function of the p-vector x and the real variable t such that

$$(8)\quad \begin{cases} \text{(i) for } \Delta x^{(j)} \geqq 0,\ j = 1, 2, \cdots, p, \text{ the mixed difference } \Delta_x G(x; t) \geqq 0 \\ \qquad \text{for every } t, \\ \text{(ii) } \int_{R(p)} d_x G(x; t) < \infty, \\ \text{(iii) for any } t,\ G(x; t) \text{ assigns zero measure to the point } x = 0 \text{ of } R(p), \\ \text{(iv) } G \text{ is continuous in } t \text{ for all } x, \\ \text{(v) if } t_1 < t_2 \text{ and } \Delta x^{(j)} \geqq 0,\ j = 1, 2, \cdots, p, \text{ then } [\Delta_x G(x; t)]_{t_1}^{t_2} \geqq 0. \end{cases}$$

LEMMA 2. *Let $\Sigma(t) = \{\sigma^{ij}(t)\}$ be a $p \times p$ symmetric matric function of t, defined and bounded for all t in a closed interval T and such that $\Sigma(t_2) - \Sigma(t_1)$ is nonnegative definite for any $t_1 < t_2$ (in T). Then all elements of $\Sigma(t)$ are of* b.v.*(bounded variation) in T.*

Hence, if $\mu(t)$ is of b.v., so is $\psi(u; t)$ of b.v. in t for every u.

NOTATION. In what follows, $A(t) = \{\alpha^{ij}(t)\}$ being a $p \times p$ matrix, we write $\mathrm{M}: dA(v)\mu(v)$ for the column-vector whose ith element is

$$\sum_{l=1}^{p} \int_{t_0}^{t} \mu^{(l)}(v)\, d\alpha^{il}(v).$$

Similarly we write $\int_{t_0}^{t} A(v)\, d\Sigma(v) A'(v)$ for the matrix whose (i, j)th element is

$$\sum_{l=1}^{p} \sum_{m=1}^{p} \int_{t_0}^{t} \alpha^{il}(v) \alpha^{jm}(v)\, d\sigma^{lm}(v).$$

By a partition $\Pi(t_0, t_l; l; m)$ of $[t_0, t_l]$, we mean a finite number of t-values

$$t_0 = t_{l,m,0} < t_{l,m,1} < \cdots < t_{l,m,n(l;m)} = t_l, \qquad t'_{l,m,j},\ j = 1, 2, \cdots, n(l; m),$$

where $t_{l,m,j-1} \leqq t'_{l,m,j} \leqq t_{l,m,j}$. The norm of the partition is $\delta(l; m) = \max_j (t_{l,m,j} - t_{l,m,j-1})$.

THEOREM 2. *Let $Z(v)$ be a p-dimensional additive process, defined and continuous in probability for $t \geqq t_0$, whose* l.c.f. *is given by* (7). *Let $A(t)$ be a real-valued $p \times p$ matrix whose elements are continuous and of* b.v. *in the closed interval $[t_0, T]$. Given any positive integer N and any values $t_1 < t_2 < \cdots < t_N$ of t in $[t_0, T]$, let $\{\Pi_m\} = \{\Pi(t_0, t_l; l; m); l = 1, 2, \cdots, N; m = 1, 2, 3, \cdots\}$ be a sequence* *of sets of partitions of the set of intervals* $\{[t_0, t_l], l = 1, 2, \cdots, N\}$ *whose norms* $\delta(l; m) \to 0$ *for* $l = 1, 2, \cdots, N$, *as* $m \to \infty$. *For any* m, *let*

$$(9)\quad S(t_0, t_l; l; m) = \sum_{j=1}^{n(l;m)} A(t'_{l,m,j})\{Z(t_{l,m,j}) - Z(t_{l,m,j-1})\}, \qquad l = 1, 2, \cdots, N.$$

Then, as $m \to \infty$,

(a) *the sequence of sets or* r.v. $\{S(t_0, t_l; l; m), l = 1, 2, \cdots, N; m = 1, 2, \cdots\}$ *converges in distribution to a set of* r.v.

$$(10)\qquad Z(A; t_0, t_l), \qquad l = 1, 2, \cdots, N,$$

whose distribution is independent of $\{\Pi_m\}$;

(b) *the* l.c.f. *of* $Z(A; t_0, t)$ *is*

$$(11)\quad \begin{aligned} \Psi(t_0, t) = iu'\left\{[A(v)\mu(v)]_{t_0}^{t} - \int_{t_0}^{t} dA(v)\mu(v)\right\} - \tfrac{1}{2}u' \int_{t_0}^{t} A(v)\, d\Sigma(v)A'(v)u \\ + \int_{R(p)\times[t_0, t]} \{e^{iu'A(v)x} - 1 - iu'A(v)x(1 + x'x)^{-1}\} \\ \cdot(1 + x'x)(x'x^{-1}\, dG(x; v); \end{aligned}$$

(c) *the joint distribution of the set of* r.v. (10) *is the same as that of the set*

$$(12)\qquad \sum_{j=1}^{l} Z(A; t_{j-1}, t_j), \qquad l = 1, 2, \cdots, N,$$

where the $Z(A; t_{j-1}, t_j)$, $j = 1, 2, \cdots, N$, *are mutually independent and the* l.c.f. *of* $Z(A; t_{j-1}; t_j)$ *is* $\Psi(t_{j-1}, t_j)$.

PROOF. First taking $N = 1$, we show that the sequence of approximating sums (9) converges in distribution to a vector r.v. whose l.c.f. is $\Psi(t_0, t_1)$. For simplicity, we shall drop the suffix m from the t-values, and also write $f(u, x; v)$ for the integrand in the third term of $\Psi(t_0, t_1)$, T for the interval $[t_0, t_1]$ and T_j for the interval $[t_{1,j-1}, t_{1,j}]$.

Since the $Z(t_{1,j}) - Z(t_{1,j-1})$, $j = 1, 2, \cdots, n(l; m)$, are mutually independent,

$$(13)\quad \begin{aligned} &\log E\{e^{iu'S}(t_0, t_1; 1; m)\} \\ &= \sum_{j=1}^{n} [\psi\{A'(t'_{1,j})u; t_{1,j}\} - \psi\{A'(t'_{1,j})u; t_{1,j-1}\}] \\ &= i\sum_{j=1}^{n} u'A(t'_{1,j})\{\mu(t_{1,j}) - \mu(t_{1,j-1})\} \\ &\quad - \frac{1}{2}\sum_{j=1}^{n} u'A(t'_{1,j})\{\Sigma(t_{1,j}) - \Sigma(t_{1,j-1})\}A'(t'_{1,j})u \\ &\quad + \sum_{j=1}^{n} \int_{R(p)} f(u, x; t'_{1,j})\, d_x\{G)x; t_{1,j}) - G(x; t_{1,j-1})\} \\ &= I_1(m) + I_2(m) + I_3(m), \qquad \textit{say}. \end{aligned}$$

It is easily seen that, given any positive U, $I_1(m)$ converges to the first term of $\Psi(t_0, t_1)$, the convergence being uniform for all u such that $u'u \leqq U$. Similarly, $I_2(m)$ converges uniformly to the second term of $\Psi(t_0, t_1)$. Finally,

$$I_3(m) = \sum_{j=1}^{n} \int_{R(p)\times T_j} f(u, x; t'_{1,j})\, d_{xv}\, G(x; v).$$

Now, let S_c be the set of all x such that $x'x \leqq c$, and let $S'_c = R(p) - S_c$. For $v \varepsilon T$, the elements of $A(v)$ are bounded in absolute value, by K say. Hence, for $u'u \leqq U$, $x \varepsilon S'_c$ and $v \varepsilon T$, we have

$$\begin{aligned} | f(u, x; v) | &= | e^{iu'A(v)x} - 1 - iu'A(v)x(1 + x'x)^{-1} | \, | (1 + x'x)(x'x)^{-1} | \\ &\leqq 2(1 + c^{-2}) + p^2 K U^{\frac{1}{2}} c^{-\frac{1}{2}} \\ &\leqq 5 \qquad\qquad c \geqq c_0 = \max(1, p^4 K^2 U). \end{aligned}$$

From (8), we know that $\int_{S'_c \times T} d_{xv} G(x; v) < \epsilon$ for $c > c_\epsilon$. Let us take $c = \max(c_0, c_\epsilon)$ and write S for S_c. Then

$$\sum_{j=1}^{n} \int_{S' \times T_j} | f(u, x; t'_{1,j}) | \, d_{xv}\, G(x; v) < 5\epsilon,$$

and $\int_{S' \times T} | f(u, x; v) | \, d_{xv} G(x; v) < 5\epsilon$. Hence,

$$\left| I_3(m) - \int_{R(p)\times T} f(u, x; v)\, d_{xv}\, G(x; v) \right|$$

$$< \sum_{j=1}^{n} \int_{S \times T_j} | f(u, x; t'_{1,j}) - f(u, x; v | \, d_{xv}\, G(x; v) + 10\epsilon.$$

From considerations of uniform continuity, it can be seen that the first term on the right-hand side is of order ϵ for sufficiently small $\delta(1; m)$ and all u such that $u'u \leqq U$. Consequently, $I_3(m)$ converges uniformly, for all u such that $u'u \leqq U$, to the third term of $\Psi(t_0, t_1)$. By the continuity theorem for characteristic functions, it follows that $\{S(t_0, t_1, 1; m), m = 1, 2, \cdots\}$ converges in distribution to a random vector $Z(A; t_0, t_1)$ whose l.c.f. is $\Psi(t_0, t_1)$ and hence independent of the sequence of partitions.

To prove (c), it is enough to indicate the proof for $N = 2$. Let $t^*_{2,m}$ be the largest $t_{2,m,j} \leqq t_1$, $j = 1, 2, \cdots, n(2; m)$. Then

$$t^*_{2,m} \to t_1; \tag{14}$$

$$S(t_0, t_2; 2; m) = S(t_0, t^*_{2,m}; 2; m) + S(t^*_{2,m}, t_2; 2; m). \tag{15}$$

The two terms on the right-hand side of (15) are mutually independent vector r.v. which, on account of (14), converge in distribution respectively to $Z(A; t_0, t_1)$ and $Z(A; t_1, t_2)$.

Now, let $\Pi^*(t_0, t_1; m)$ be the partition obtained by superposing $\Pi(t_0, t_1; 1; m)$ on $\Pi(t_0, t^*_{2,m}; 2; m)$ and let $S^*(t_0, t_1; m)$ be the corresponding approximating sum. By making use of the fact that $A(t)$ is of b.v. and that $Z(t)$ is uniformly continuous in probability in every closed interval in which it is pointwise continuous, we can show that

$$S^*(t_0, t_1; m) - S(t_0, t_1; 1; m) \quad \text{and} \quad S^*(t_0, t_1; m) - S(t_0, t^*_{2,m}; 2; m)$$

both converge in probability to zero. Hence so does

$$S(t_0, t_1; 1; m) - S(t_0, t^*_{2,m}; 2; m).$$

But if X_m, Y_m are mutually independent and converge in distribution to X, Y, and $0_m \to 0$ in probability, then $\{X_m + 0_m, X_m + Y_m\}$ converges in distribution to $(X, X + Y)$, X and Y being mutually independent. Therefore, the set of r.v. $\{S(T_0, t_1; 1; m), S(t_0, t_2; 2; m)\}$ converges in distribution to $\{Z(A; t_0, t_1), Z(A; t_0, t_1) + Z(A; t_1, t_2)\}$, where $Z(A; t_0, t_1)$ and $Z(A; t_1, t_2)$ are independent.

REMARKS. The fact that $A(t)$ is of b.v. has been used in the proof only in establishing the convergence of the first term in (13). Therefore, if $\mu(t)$ happens to be itself of b.v., the theorem certainly holds for any continuous $A(t)$.

Incidentally, we notice that if $p = 1$, $A(t) = t$ and $G(x; t) = G_1(x)G_2(t)$, where $G_2(t)$ is a distribution function on $R(1)$ with a finite second moment, and if $\mu(t)$ and $\Sigma(t)$ are such that the first two terms of (11) have finite limits as $t_0 \to -\infty$ and $t \to \infty$, then the limit of the l.c.f. (11) is a g-function of Kallianpur and Robbins [5].

The definition of $Z(A; t_0, t)$ as the limit-in-distribution of a sequence of approximating sums suggests the formal equation

$$Z(A; t_0, t) = {}_D\int_{t_0}^{t} A(v)\, dZ(v). \tag{16}$$

The D before the integral sign is a reminder of the fact that this is an "integral-in-distribution." We note that for $t_0 < t_1 < t_2$, the distribution of ${}_D\int_{t_0}^{t_2} A(v)\, dZ(v)$ is the convolution of the distributions of ${}_D\int_{t_0}^{t_1} A(v)\, dZ(v)$ and ${}_D\int_{t_1}^{t^2} A(v)\, dZ(v)$. In other words, for fixed t_0 and variable $t \geqq t_0$, the random function ${}_D\int_{t_0}^{t} A(v)\, dZ(v)$ is an additive process. Its l.c.f. $\Psi(t_0, t)$ should, therefore, be capable of being displayed in the general form (7). This was actually found to be possible with some restrictions on $A(t)$; how it can be done in general is not known. Anyway, the resulting expression seems too cumbrous for use.

It may also be noted that the formal equation (16) actually represents the true relation between the l.c.f.'s of its two members. The increments of $Z(t)$ are mutually independent and the l.c.f. of $A(v)\, dZ(v)$ is $[d_t\psi\{A'(v)u;\, t\}]_{t=v}$. Thus we expect, by an extension of the law concerning the l.c.f. of a convolution of finite order, that the l.c.f. of the right-hand side of (16) is

$$\int_{t_0 \leqq v \leqq t} [d_t\, \psi\{A'(v)u;\, t\}]_{t=v},$$

if at all such a thing exists. Actually, this is the same as $\Psi(t_0, t)$ in (11).

In allowing $Z(t)$ the freedom of the wide class of additive processes, we have restricted ourselves necessarily to a rather weak limiting process for the definition of the integral, namely the limit-in-distribution. Stronger definitions of the integral have already been in use for some time. For instance, if $Z(t)$ has finite covariance and orthogonal increments, the integral defined as a limit in the mean is a random variable which with probability 1 is uniquely determined. Since there are additive processes which do not have a finite covariance, this definition of the integral does not suit our purpose.

Now, let $A_1(t)$ and $A_2(t)$ be continuous and of b.v. in an interval $[t_0, T]$; then so is $A_2(t)A_1(t)$, and hence

$$Z(A_2 A_1; t_0, t) = {}_D\int_{t_0}^{t} A_2(v)A_1(v)\, dZ(v), \qquad t_0 \leqq t \leqq T$$

exists and, for fixed t_0 and variable t, is an additive process. So also

$$Z(A_2, A_1; t_0, t) = {}_D\int_{t_0}^{t} A_2(v)\, d_v Z(A_1; t_0, v)$$

is an additive process.

NOTATION. For $X(t)$ and $Y(t)$, two random functions defined for all t in a dominant T, we shall write $X(t) \overset{D}{=} Y(t)$ to imply that, given any n and any $t_1, t_2, \cdots, t_n$ in T, the joint distribution of $X(t_1), \cdots, X(t_n)$ is the same as that of $Y(t_1), \cdots, Y(t_n)$.

THEOREM 3. *If $A_1(t)$ and $A_2(t)$ are continuous and of* b.v. *in* $[t_0, T]$, *then*

$$Z(A_2, A_1; t_0, t) \overset{D}{=} Z(A_2 A_1; t_0, t)$$

for all t in $[t_0, T]$.

PROOF. Given $t_1 < t_2 < \cdots < t_n$, let $Y(j) = Z(A_2A_1; t_0, t_j) - Z(A_2A_1; t_0, t_{j-1})$, and let $\psi_j(u)$ be the l.c.f. of $Z(A_2A_1; t_0, t_j)$, so that the l.c.f. of $Y(j)$ is $\psi_j(u) - \psi_{j-1}(u)$. Then

$$\begin{aligned}\text{l.c.f. } \{Z(A_2 A_1; t_0, t_j), j = 1, \cdots, n\} &= \log E\left\{\exp\left[i\sum_{j=1}^{n} u'(j)Z(A_2 A_1; t_0, t_j)\right]\right\}\\ &= \log E\left\{\exp\left[i\sum_{j=1}^{n}\sum_{k=j}^{n} u'(k)Y(j)\right]\right\}\\ &= \sum_{j=1}^{n} \psi_j\left\{\sum_{k=j}^{n} u(k)\right\} - \sum_{j=1}^{n} \psi_{j-1}\left\{\sum_{k=j}^{n} u(k)\right\}.\end{aligned}$$

In the case of $Z(A_2, A_1; t_0, t)$, we have a similar expression in terms of the l.c.f. of $Z(A_2, A_1; t_0, t_j)$. Hence, it is enough to show that, for any t, the l.c.f.'s of $Z(A_2A_1; t_0, t)$ and $Z(A_2, A_1; t_0, t)$ are the same. For this, we can derive the l.c.f. of $Z(A_2, A_1; t_0, t)$ by the procedure used in Theorem 2, and then reduce it to the form (11) with $A(v) = A_2(v)A_1(v)$, which is the l.c.f. of $Z(A_2A_1; t_0, t)$.

Hence the distributions of $Z(A_2, A_1; t_0, t)$ are the same as those of $Z(A_2A_1; t_0t)$; that is to say, the formal equation

$$\int_{D\,t_0}^{t} A_2(v)\, d_v \int_{D\,t_0}^{v} A_1(u)\, dZ(u) \overset{D}{=} \int_{D\,t_0}^{t} A_2(v) A_1(v)\, dZ(v)$$

is rigorously true.

COROLLARY 2. *Given any* $Z(t)$ *and* $A(t)$ *satisfying the conditions of Theorem* 2 *and the additional condition that the determinant* $|A(t)|$ *be bounded away from zero in* $[t_0, T]$, *there exists an additive process* $Z^*(t)$ *such that*

$$Z(t) - Z(t_0) \overset{D}{=} \int_{D\,t_0}^{t} A(v)\, dZ^*(v).$$

In fact, $Z^*(t) \overset{D}{=} Z(A^{-1}; t_0, t)$.

PROOF. The result follows immediately from the fact that, since the determinant and minor determinants of $A(t)$ are of b.v. and $|A(t)|$ is bounded away from zero, $A^{-1}(t)$ is of b.v.

3. Exponential variation. Going back to the $X(t)$ with which we started, we have the following result.

THEOREM 4. *If* $X(t)$ *and* $A(h)$ *satisfy the assumptions of Theorem* 1, *then there exist an additive process* $Z(t)$ *and a constant matrix* Λ *such that*

$$X(t) \overset{D}{=} e^{(t-t_0)\Lambda} X(t_0) + \int_{D\,t_0}^{t} e^{(t-v)\Lambda}\, dZ(v), \qquad t \geqq t_0. \tag{17}$$

Conversely, if Λ *is any constant matrix,* $Z(t)$ *any additive process, and* $X(t_0)$ *a* r.v. *independent of all* $Z(t+h) - Z(t)$ *for* $t \geqq t_0$ *and* $h > 0$, *then an* $X(t)$ *satisfying* (17) *has the property that* $X(t+h) - e^{h\Lambda}X(t)$ *is independent of all* $X(t')$ *for* $t_0 \leqq t' \leqq t$.

PROOF. From Theorem 1, we know that $X(t) = e^{\Lambda t}X^*(t)$, where $X^*(t)$ is an additive process; and since $e^{-\Lambda t}$ has the properties of the $A(t)$ of Corollary 2, we have

$$X^*(t) - X^*(t_0) \overset{D}{=} \int_{D\,t_0}^{t} e^{-\Lambda v}\, dZ(v).$$

This immediately gives (17); the converse is obvious.

REMARKS. Random functions of the nature of (17) arise in connection with certain physical processes. For instance, Loève [10] has considered the problem of a reservoir of water which is losing its contents exponentially and gaining from random precipitation. In this case, $Z(t)$ is a Poisson process with jumps of variable magnitude. On account of such applications and equation (17), one may refer to $X(t)$ as the result of exponential variation of an additive process. Now, it may happen that such an $X(t)$ itself undergoes an exponential variation. We shall now deal with the result of such an iterated exponential variation.

THEOREM 5. *Let* $X_1(t)$ *be a random function satisfying the assumptions of Theorem* 1, *and consequently such that*

$$X_1(t) \overset{D}{=} e^{(t-t_0)\Lambda_1} X_1(t_0) + \int_{D\,t_0}^{t} e^{(t-v)\Lambda_1}\, dZ(v). \tag{18}$$

Let Λ_2 *be a* $p \times p$ *matrix such that* $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$ *and* $|\Lambda_1 - \Lambda_2| \neq 0$. *Given any positive integer* N *and any values* $t_1 < t_2 < t_3 < \cdots < t_N$ *of* t, *with* $t_1 \geqq t_0$, *let* $\{\Pi_m\} = \{\Pi(t_0, t_l; l; m), l = 1, 2, \cdots, N; m = 1, 2, \cdots\}$ *be a sequence of sets of partitions of the set of intervals* $\{[t_0, t_l], l = 1, 2, \cdots, N\}$ *whose norms* $\delta(l; m) \to 0$ *for* $l = 1, 2, \cdots, N$, *as* $m \to \infty$. *For any m, let*

$$S(t_0, t_l; l; m) = \sum_{j=1}^{n} e^{(t_l - t'_{l,m,j})\Lambda_2}\{X_1(t_{l,m,j}) - X_1(t_{l,m,j-1})\}, \tag{19}$$
$$l = 1, 2, \cdots, N.$$

Then, as $m \to \infty$,

(a) *the set of* r.v. $\{S(t_0, t_l; l; m), l = 1, 2, \cdots, N; m = 1, 2, \cdots\}$ *converges in distribution to a set of* r.v. $\{X_2(t_l), l = 1, 2, \cdots, N\}$ *whose distribution is independent of* $\{\Pi_m\}$;

(b) *in fact, for varying t*,

$$\begin{aligned} X_2(t) \overset{D}{=} {} & \Lambda_1(\Lambda_1 - \Lambda_2)^{-1}\{e^{(t-t_0)\Lambda_1} - e^{(t-t_0)\Lambda_2}\}X_1(t_0) \\ & + {}_D\int_{t_0}^{t} \{\Lambda_1 e^{(t-v)\Lambda_1} - \Lambda_2 e^{(t-v)\Lambda_2}\}(\Lambda_1 - \Lambda_2)^{-1}\, dZ(v); \end{aligned} \tag{20}$$

(c) *Also*, $X_2(t)$ *has the property that*

$$X_2(t + 2h) - (e^{h\Lambda_1} + e^{h\Lambda_2})X_2(t + h) + e^{h(\Lambda_1+\Lambda_2)}X_2(t) \tag{21}$$

is independent of all $X_2(t')$, $t_0 \leqq t' \leqq t$.

PROOF. Let $e^{-t\Lambda_1}X_1(t) = X^*(t)$, so that $X^*(t)$ is an additive process. For convenience, we shall write $t_{l,j}$ for $t_{l,m,j}$ and $\Delta_{l,j}$ for $\{X^*(t_{l,m,j}) - X^*(t_{l,m,j-1})\}$. Then

$$\begin{aligned} S(t_0, t_l; l; m) &= \sum_{j=1}^{n(l;m)} e^{(t_l - t'_{l,j})\Lambda_2}\{e^{\Lambda_1 t_{l,j}}[X^*(t_0) + \Delta_{l,1} + \cdots + \Delta_{l,j}] \\ &\qquad - e^{\Lambda_1 t_{l,j-1}}[X^*(t_0) + \Delta_{l,1} + \cdots + \Delta_{l,j-1}]\} \\ &= \sum_{1}^{n(l;m)} A_{l,j}X^*(t_0) + \sum_{j=1}^{n(l,m)} B_{l,j}\Delta_{lj}, \end{aligned} \tag{22}$$

where

$$A_{l,j} = e^{(t_l - t'_{l,j})\Lambda_2}(e^{\Lambda_1 t_{l,j}} - e^{\Lambda_1 t_{l,j-1}}),$$

$$B_{l,j} = e^{(t_l - t'_{l,j})\Lambda_2 + \Lambda_1 t_j} + \sum_{k=j+1}^{n(l;m)} e^{(t_l - t'_{l,k})\Lambda_2}(e^{\Lambda_1 t_{l,k}} - e^{\Lambda_1 t_{l,k-1}}).$$

The two terms on the right-hand side in (22) are independent. The first converges in distribution to

$$\int_{t_0}^{t_l} e^{(t_l - v)\Lambda_2}\, d[e^{v\Lambda_1}X^*(t_0)] = a(t_l)X^*(t_0), \quad \text{say.} \tag{23}$$

By a procedure similar to that used in the proof of Theorem 2, we see that the second term on the right-hand side of (22) converges in distribution to

$$(24)\qquad {}_D\int_{t_0}^{t_l}\left\{e^{(t_l-v)\Lambda_2+v\Lambda_1}+\int_v^{t_l}e^{(t_l-v')\Lambda_2}\,de^{v'\Lambda_1}\right\}dX^*(v) = Y(t_l),\quad \text{say.}$$

From (23), (24) and Theorem 3, it follows that for any t_l, $S(t_0, t_l; l; m)$ converges in distribution to $X_2(t_l)$ as given by (20). To establish the simultaneous convergence in distribution of the set (19) to $\{X_2(t_l), l = 1, 2, \cdots, N,\}$ we can again use the procedure of the corresponding part of the proof of Theorem 2. Finally, from (b) we immediately have (c).

REMARKS. The assumption that $|\Lambda_1 - \Lambda_2| \neq 0$ is necessary only for the final reduction of $X_2(t)$ to the form (20). Without this restriction (a) is still true, and also (b), except that in (20) the right-hand side has to be replaced by $\{a(t)X^*(t_0) + Y(t)\}$ from (23) and (24); it follows that (c) holds. In fact, we have results of this type even if neither of the assumptions $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$ and $|\Lambda_1 - \Lambda_2| \neq 0$ is satisfied.

Thus we see that whereas the result of an exponential decay, defined as a limit in distribution, of an additive process is a random function satisfying a linear relation of the first order, the result of an exponential decay of such a random function satisfies a linear relation of the second order. Incidentally, we have come across a wide class of random functions with respect to which one can define Riemann-Stieltjes "integrals-in-distribution." This raises the question of characterizing the class of all random functions having this property. We have seen that this class is wider than that of random functions with independent increments; but its total content is not known.

It may be noted that equation (17) is formally the same as the solution of the set of differential equations

$$dX - \Lambda X\,dt = dZ(t), \qquad t \geqq t_0 .$$

Likewise, (20) is formally the same as the solution of the set $dX_2 - \Lambda_2 X_2\,dt = dX_1(t)$, that is,

$$d^2X_2 - (\Lambda_1 + \Lambda_2)\,dX_2\,dt + \Lambda_1\Lambda_2 X_2\,(dt)^2 = d^2Z(t).$$

In fact, when derivatives in mean square exist, the solutions of these equations are rigorously given in terms of integrals in the mean. We have now seen that the formalism holds in terms of integrals in distribution, when we have on the right-hand side of the equations random functions of certain types.

4. Acknowledgement. The author wishes to express his thanks to Dr. Herbert Robbins for advice and help, and to the Institute of Statistics of the University of North Carolina, in particular to Dr. Harold Hotelling, for the liberal financial assistance and other facilities given to the author during the course of this work.

REFERENCES

[1] J. L. Doob, *Stochastic Processes*, John Wiley and Sons, 1953.
[2] W. Feller, "Neuer Beweis für die Kolmogoroff-P. Lévysche Charakterisierung," *Bull. Internat. Acad. Yougoslave Cl. Sci. Math. Phys. Tech.*, Vol. 32 (1939), pp. 106–113.
[3] S. G. Ghurye, "Random functions satisfying certain linear relations," (abstract), *Ann. Math. Stat.*, Vol. 23 (1952), p. 646.
[4] B. V. Gnedenko, "Limit theorems for sums of independent random variables," *Uspehi Matem. Nauk*, Vol. 10 (1944), pp. 115–165.
[5] G. B. Kallianpur and H. Robbins, "On a class of infinitely divisible distributions," (abstract), *Ann. Math. Stat.*, Vol. 23 (1952), p. 146.
[6] A. Khintchine, "Déduction nouvelle d'une formule de M. Paul Lévy," *Bull. Univ. d'état Mosc., serie internationale, Sec. A*, Vol. 1 (1937), pp. 1–5.
[7] Paul Lévy, *Theorie de l'Addition des Variables Aléatoires*, Gauthier-Villars, Paris, 1937.
[8] Paul Lévy, *Processus Stochastiques et Mouvement Brownien*, Gauthier-Villars, Paris, 1948.
[9] Paul Lévy, "Random functions: General theory with special reference to Laplacian random functions," *Univ. California Publ. Stat.*, Vol. 1 (1953), pp. 331–390.
[10] M. Loève, "Schema des debits," (unpublished, written in 1944).
[11] B. von Sz. Nagy, "Über messbare Darstellungen Liescher Gruppen," *Math. Ann.*, Vol. 112 (1931), pp. 286–296.