ON CERTAIN CONFIDENCE CONTOURS FOR DISTRIBUTION FUNCTIONS

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0. Summary. By a generalisation of a theorem by Doob, certain confidence or acceptance contours for distribution functions are obtained. The power of tests based on such contours is briefly discussed and some approximate results derived. Using the aforementioned generalisation of Doob's theorem, the limiting joint probability distribution of the coordinates of the maximum deviation between a sample distribution and the corresponding parent distribution is evaluated.

1. Introduction. Let $F_N^*(U)$ be the empirical distribution function for a sample of N mutually independent observations on a statistical variate with a continuous distribution function $F(U)$. Consider the process

$$X_N(U) = [F_N^*(U) - F(U)]\sqrt{N}.$$

Given $F_N^*(U)$, the probabilities for events such as

$$\text{(A)} \qquad G_1[F(U)] \leqq X_N(U) \leqq G_2[F(U)],$$

for all U, can be used for testing the hypothesis that $F(U)$ is a given function. The situation is illustrated in Fig. 1. The interval AB is called an acceptance interval, the interval CD is a confidence interval. Allowing U to vary, acceptance and confidence regions are obtained (cf. [3], p. 515; [13]).

Probabilities for an event such as (A) have the attractive property of being independent of the distribution function $F(U)$. One may suppose that $F(U) = U$, with $0 \leqq U \leqq 1$.

The limiting distributions

$$\text{(B)} \quad \lim_{N\to\infty} P_N\{X_N(U) \leqq G_2(U), 0 \leqq U \leqq 1\} \qquad \text{(one-sided alternative)},$$

$$\text{(C)} \quad \lim_{N\to\infty} P_N\{G_1(U) \leqq X_N(U) \leqq G_2(U), 0 \leqq U \leqq 1\} \qquad \text{(two-sided alternative)},$$

where the inequalities within brackets should be fulfilled for every U, have been derived by Kolmogorov [7] for $G_1(U) = -a$ and $G_2(U) = a$. Related problems have been considered by Smirnov [12].

Doob [5] has demonstrated the Kolmogorov and Smirnov theorems by replacing the process $X_N(U)$ by a Gaussian process $X(U)$ with the same correlation function. This transformation has been justified by Donsker [4]. The process $X(U)$ is then transformed to a Wiener process $W(t)$, and limiting probabilities concerning $X_N(U)$ are transformed to probabilities concerning $W(t)$.

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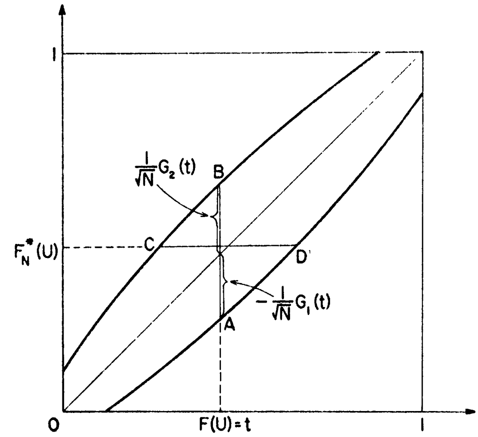

FIG. 1

For finite N, probability distributions involving confidence contours $G(U) = a$ have been tabulated by Massey [10], [11] and Birnbaum and Tingey [2]. A general method for calculating distributions (C) above has been developed by Anderson and Darling [1], (cf. [6]).

For a given U, $F_N(U)$ follows the binomial distribution and we have, supposing $F(U) = U$ with $0 \leqq U \leqq 1$,

$$E\{X_N(U)\} = 0, \qquad E\{X_N(U_1)X_N(U_2)\} = U_1(1 - U_2), \qquad 0 < U_1 \leqq U_2 < 1.$$

In particular, $E\{X_N^2(U)\} = U(1 - U)$ for $0 < U < 1$. This shows that the variance decreases towards the tails of the distribution. Thus, in constructing the contours $G_1(U) \leqq X_N(U) \leqq G_2(U)$ it seems reasonable to let the width $G_2(U) - G_1(U)$ decrease towards the ends of the distribution (cf. section 5). In absence of general principles in this respect, the form $-G_1(U) = G_2(U) = a\sqrt{U(1 - U)}$ with exclusion of the points $U = 0$ and $U = 1$ has been suggested by Anderson and Darling [1].

If, for example,

$$\text{(D)} \qquad G_2(U) = \begin{cases} (a - b)U + b & 0 \leqq U \leqq \frac{1}{2}, \\ (b - a)U + a & \frac{1}{2} \leqq U \leqq 1, \end{cases} \qquad a > b$$

then deviations of $X_N(U)$ at the extremes of the distribution will have a greater chance of being detected. Or, the width of the confidence contours can be made smaller at the tails than at the middle of the distribution. Naturally, here also general principles for the choice of a and b are lacking.

We may also be interested in the deviations $X_N(U)$ for a certain part of the distribution $F(U)$. That is, $G_2(U)$ should for example be of the form

$$\text{(E)} \qquad G_2(U) = \begin{cases} aU + b & A \leqq U \leqq B \\ (1 - U)\sqrt{N} & \text{elsewhere.} \end{cases}$$

In Section 3 below the limiting probabilities for inequalities of type (B) are evaluated for the form of $G(U)$ indicated by (D) and (E). For the derivation of the probabilities in Section 3 we use generalisations of theorems proved by Doob [5] concerning the Wiener process. These generalisations, proved in Section 2, follow very simply from the symmetric property of the aforementioned Gaussian process $X(U)$. The theorem proved in Section 2 is in agreement with the fact that the Wiener process is continuous with probability one and may be approximated by a discrete Gaussian process with a corresponding correlation matrix.

In Section 4, a numerical example is given of upper and lower limits for the power function in the case when one of the earlier derived limiting probability distributions is used for testing a specified normal distribution against a certain other normal distribution. The limits, which are capable of improvement, are rather wide and indicate for the example chosen a relative power of about 60 per cent, compared with the most powerful test. This comparatively low value is not surprising, considering the general nature of the testing procedure used.

From the aforementioned properties of the $X_N(U)$ process it follows that a large deviation between the empirical and theoretical distribution functions $F_N(U)$ and $F(U)$ is more probable in the middle of the distribution than at the extremes. It may therefore be of interest to consider both coordinates of the maximum of $X_N(U)$. The joint and conditional distributions involved are derived in Section 5. (Corresponding expressions for $W(t)$ are given by Lévy [8], chap. 6.)

2. Generalisation of a theorem by Doob. Let $W(t)$, $0 < t < \infty$, be a Gaussian process for which

$$\Pr\{W(0) = 0\} = 1, \qquad E\{W(t)\} = 0, \qquad E\{W(s)W(t)\} = s, \quad s \leqq t.$$

The process $W(t)$, called the Brownian movement process or the (normalised) Wiener process, has uncorrelated, and thus independent, increments.

For the Gaussian process $W_{t_0}(t')$ defined for $t' = t - t_0$ by

$$W_{t_0}(t') = W(t) - W(t_0), \qquad t \geqq t_0,$$

we have $E\{W_{t_0}(t')\} = 0$. Further, for $t_1' \leqq t_2'$,

$$\begin{aligned} E\{W_{t_0}(t_1')W_{t_0}(t_2')\} &= E\{[W(t_1' + t_0) - W(t_0)][W(t_2' + t_0) - W(t_0)]\} \\ &= t_1' + t_0 - t_0 - t_0 + t_0 = t_1' \end{aligned}$$

Thus, $W_{t_0}(t')$ is also a Wiener process.

It has been proved by Doob [5] that

$$\text{(1)} \qquad \Pr\{W(t) \leqq at + b\} = 1 - e^{-2ab}, \qquad a \geqq 0, b > 0; \text{ all } t,\ 0 \leqq t < \infty.$$

The inequality within brackets must be fulfilled for every t in the given interval. The following generalisation will be proved.

THEOREM 1: *Let the process $W(t)$ pass through the points $\{x, s_1\}$ and $\{y, s_2\}$, with $x \leqq y$. Then*

$$\Pr\{W(t) \leqq at + b, x \leqq t \leqq y \mid W(x) = s_1, W(y) = s_2\} = 1 - \exp\left\{-\frac{2R}{1 - R^2}\frac{P_1 - s_1}{\sqrt{x}}\frac{P_2 - s_2}{\sqrt{y}}\right\}$$

where $R = \sqrt{x/y}$, and

$$s_1 \leqq P_1 = ax + b, \qquad s_2 \leqq P_2 = ay + b.$$

For the proof, we shall use the following transformations due to Doob [5]. First, a process $X(U)$ is defined by

$$X(U) = \frac{1}{t+1} W(t), \quad 0 \leqq t < \infty; \qquad U = \frac{1}{1+t}, \quad 0 \leqq U \leqq 1. \tag{2}$$

For the Gaussian process $X(U)$ we then have

$$E\{X(U)\} = 0, \qquad E\{X(U)X(V)\} = U(1 - V), \qquad 0 \leqq U \leqq V \leqq 1.$$

Further, if $U_1 = 1 - U$ and $V_1 = 1 - V$, then $E\{X(U_1)X(V_1)\} = E\{X(U)X(V)\}$. Making use of this symmetric property of the $X(U)$ process, we have

$$\Pr\{X(U) \leqq f(U), 0 < U \leqq U' \mid X(U') = x'\} = \Pr\{X(U) \leqq f(1 - U), 1 - U' \leqq U < 1 \mid X(1 - U') = x'\},$$

where $f(U') \geqq x'$. By applying the transformation (2) we obtain

$$\begin{aligned} &\Pr\left\{W(t) \leqq (t + 1)f\left(\frac{t}{1+t}\right), \quad 0 < t \leqq t' \mid W(t') = s_1\right\} \\ &\quad = \Pr\left\{W(t) \leqq (t + 1)f\left(\frac{1}{1+t}\right), \quad \frac{1}{t'} \leqq t < \infty \;\middle|\; W\left(\frac{1}{t'}\right) = \frac{s_1}{t'}\right\} \end{aligned} \tag{3}$$

where $t' = U'/(1 - U')$ and $s_1 = x'(t' + 1)$. In the case when $f(t) = (a - b)t + b$, we have

$$(t + 1)f\left(\frac{t}{1+t}\right) = at + b; \qquad (t + 1)f\left(\frac{1}{1+t}\right) = bt + a,$$

and

$$\begin{aligned} &\Pr\{W(t) \leqq at + b, 0 < t \leqq t' \mid W(t') = s_1\} \\ &\quad = \Pr\left\{W(t) \leqq bt + a, \frac{1}{t'} \leqq t < \infty \;\middle|\; W\left(\frac{1}{t'}\right) = \frac{s_1}{t'}\right\} \end{aligned} \tag{4}$$

Using the fact that $W(t) - W(1/t')$ is also a Wiener process, we further have

$$\Pr\{W(t) \leqq bt + a, 1/t' \leqq t < \infty \mid W(1/t') = s_1/t'\} = \Pr\{W(t) \leqq bt + b/t' + a - s_1/t', 0 < t < \infty\}.$$

Thus from (1), (3), and (4),

$$\Pr\{W(t) \leqq at + b,\quad 0 < t \leqq t' \mid W(t') = s_1\}$$

$$(5)\qquad = \Pr\left\{W(t) \leqq bt + \frac{at' + b - s_1}{t'},\quad 0 < t < \infty\right\}$$

$$= 1 - \exp\left\{-2b\,\frac{P_1 - s_1}{t'}\right\},\quad s_1 \leqq P_1 = at' + b.$$

Further, according to the above mentioned property of the $W(t)$ process

$$\Pr\{W(t) \leqq at + b, x \leqq t \leqq y \mid W(x) = s_1', W(y) = s_2\}$$

$$= \Pr\{W(t) \leqq at + ax + b - s_1',\quad 0 < t \leqq y - x \mid W(y - x) = s_2 - s_1'\}.$$

Finally from (5), taking $t' = y - x$ and $s_1 = s_2 - s_1'$,

$$\Pr\{W(t) \leqq at + b,\quad x \leqq t \leqq y \mid W(x) = s_1,\quad W(y) = s_2\}$$

$$(6)\qquad = 1 - \exp\{-2(ax + b - s_1)(P_2 - s_2)/(y - x)\}$$

$$= 1 - \exp\left\{-2\,\frac{P_1 - s_1}{\sqrt{x}}\,\frac{P_2 - s_2}{\sqrt{y}}\,\frac{R}{1 - R^2}\right\};$$

where $R = \sqrt{x/y}$ with $P_1 = ax + b \geqq s_1$ and $P_2 = ay + b \geqq s_2$.

The same method can be used to generalise the result in [5] concerning two-sided probabilities. The following result, obtained by using formula (4.3), p. 398, of Doob [5], will be given without proof:

$$P'' = \Pr\{|\,W(t)\,| \leqq at + b\,|,\quad x \leqq t \leqq y \mid W(x) = s_1,\quad W(y) = s_2\}$$

$$= 1 - \sum_{m=1}^{\infty}\left(\exp\left\{\frac{-2R}{(1 - R^2)\sqrt{xy}}\,[(2m - 1)P_1 - s_1][(2m - 1)P_2 - s_2]\right\}\right.$$

$$(7)\qquad + \exp\left\{\frac{-2R}{(1 - R^2)\sqrt{xy}}\,[(2m - 1)P_1 + s_1][(2m - 1)P_2 + s_2]\right\}$$

$$- \exp\frac{-2R}{(1 - R^2)\sqrt{xy}}\,[(2mP_1 - s_1)(2mP_2 + s_2) + s_1 s_2]\Big\}$$

$$\left.- \exp\frac{-2R}{(1 - R^2)\sqrt{xy}}\,[(2mP_1 + s_1)(2mP_2 - s_2) + s_1 s_2]\Big\}\right)$$

with $R = \sqrt{x/y}$ while $P_1 = ax + b$ and $P_2 = ay + b$.

Finally, a remark concerning the one-sided and two-sided alternatives. Let a be the event that $W(t)$ passes $G_1(t)$ and b the event that $W(t)$ passes $G_2(t)$. Then

$$P\{G_1(t) \leqq W(t) \leqq G_2(t)\} = 1 - P\{a + b\} = 1 - P(a) - P(b) + P(ab)$$

According to the correlation properties of the $W(t)$ process, we have $P(ab) < P(a)P(b)$. Then for small values of $P(a)$ and $P(b)$,

$$1 - P(a + b) \simeq 1 - P(a) - P(b)$$

Even if $P(a)$ and $P(b)$ are moderately small, $P(ab)$ may be neglected. In that case the probability for a two-sided alternative may be computed from the probabilities for the one-sided alternatives involved.

3. Some examples. We shall now consider some examples of special interest concerning the use of the relations in Section 2.

A. First we consider

$$P_1 = P\left\{W(t) \begin{array}{l} \leqq at + b; 0 < t \leqq 1 \\ \leqq bt + a; 1 \leqq t < \infty \end{array}\right\}.$$

Using (4) and (5), we get

$$(8)\qquad \begin{aligned} P_1 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a+b} [1 - e^{-2b(a+b-s_1)}]^2 e^{-s_1^2/2}\, ds_1 \\ &= \Phi(a + b) - 2e^{-2ab}\Phi(a - b) + e^{-4b(a-b)}\Phi(a - 3b) \end{aligned}$$

where $\Phi(x)$ is the normal (cumulative) distribution function with mean zero and unit variance. Further, using the transformation (2) in Section 2 we have

$$P_1 = P\left\{X(U) \begin{array}{l} \leqq (a - b)U + b; 0 < U \leqq \frac{1}{2} \\ \leqq (b - a)U + a; \frac{1}{2} \leqq U < 1 \end{array}\right\}.$$

Then, replacing $X(U)$ by $X_N(U)$ as indicated in Section 2, for $F(U^*) = \frac{1}{2}$,

$$P_1 = \lim P_N\left\{[F_N^*(U) - F(U)]\sqrt{N} \begin{array}{l} \leqq (a - b)F(U) + b; -\infty < U \leqq U' \\ \leqq (b - a)F(U) + a; U' \leqq U < \infty \end{array}\right\}.$$

An expression of the above type for $a > b$ gives greater weight to deviations at the extremes of $F(U)$ than does the ordinary expression with $a = b$.

B. Next we consider

$$\begin{aligned} \mathrm{P}_2 &= \Pr\left\{W(t) \begin{array}{l} \leqq at + b; 0 < t \leqq 1; \\ \geqq -(at + b); 1 \leqq t < \infty; \end{array} \quad b > 0; a + b > 0\right\} \\ &= \lim P_N\left\{[F_N^*(U) - F(U)]\sqrt{N} \begin{array}{l} \leqq (a - b)F(U) + b; -\infty < U \leqq U^* \\ \geqq -(a - b)F(U) - b; U^* \leqq U < \infty \end{array}\right\}, \end{aligned}$$

where, as before, $F(U^*) = \frac{1}{2}$. We have

$$\begin{aligned} P_2 &= \frac{1}{\sqrt{2\pi}} \int_{-(a+b)}^{a+b} [1 - e^{-2b(a+b-s_1)}] e^{-s_1^2/2} [1 - e^{-2a(a+b+s_1)}]\, ds_1 \\ &= 1 - 2\Phi(-a - b) - e^{-2ab}[1 - 2\Phi(-a - 3b)] + e^{-8ab}[\Phi(3a - b) - \Phi(a - 3b)] \end{aligned}$$

Probabilities of this type can be used, for example, in cases when the dispersion in $F^*(U)$ is of particular interest.

C. Finally we consider

$$P_3 = \Pr\{\,|W(t)| \leqq at + b;\, x \leqq t \leqq y\}$$
$$= \lim P_N\{\,|F_N^*(U) - F(U)|\sqrt{N} \leqq (a - b)F(U) + b,\ U' \leqq U \leqq U''\}$$

where $F(U') = x/(1 + x)$ and $F(U'') = y/(1 + y)$. From (7) we get

$$P_3 = \frac{1}{2\pi\sqrt{1 - R^2}\sqrt{xy}} \int_{-p_1}^{p_1} ds_1 \int_{-p_2}^{p_2} ds_2 P'' \cdot \exp\left\{\frac{-1}{2(1 - R^2)}\left[\left(\frac{s_1}{\sqrt{x}}\right)^2 - \frac{2Rs_1 s_2}{\sqrt{xy}} + \left(\frac{s_2}{\sqrt{y}}\right)^2\right]\right\}; \qquad \begin{matrix} p_1 = ax + b \\ p_2 = ay + b. \end{matrix}$$

With some reduction we obtain

$$P_3 = \frac{1}{2\pi\sqrt{1 - R^2}} \int_{-p_1/\sqrt{x}}^{+p_1/\sqrt{x}} ds_1 \int_{-p_2/\sqrt{y}}^{+p_2/\sqrt{y}} ds_2 \exp\left\{\frac{-1}{2(1 - R^2)}[s_1^2 - 2Rs_1 s_2 + s_2^2]\right\}$$
$$- \frac{1}{2\pi\sqrt{1 - R^2}}\, 2\sum_{m=1}^{\infty} (-1)^{m+1} e^{-2m^2ab} \int_{A_1}^{A_2} ds_1 \int_{B_1}^{B_2} ds_2 \exp\left\{\frac{-1}{2(1 - R^2)} \cdot [s_1^2 + 2Rs_1 s_2 + s_2^2]\right\},$$

where

$$\begin{matrix} A_1 = -(p_1 + 2axm)/\sqrt{x}, & B_1 = -(p_2 + 2bm)/\sqrt{y}, & \begin{cases} p_1 = ax + b, \\ p_2 = ay + b. \end{cases} \\ A_2 = (p_1 - 2axm)/\sqrt{x}, & B_2 = (p_2 - 2bm)/\sqrt{y}, & \end{matrix}$$

This distribution, for $a = b$, has been evaluated by Anderson and Darling [1] and Maniya [9].

4. Power functions. It may be of interest to compare the power of the one-sided test for a certain class of hypotheses for which the power of the most powerful test is known. As an example we choose the case where a normal distribution with, let us say, mean zero and unit variance is tested against the class of normal distributions with the same variance but a negative mean.

Let $\Phi(U)$ be the normal cumulative distribution function with mean zero and unit variance and $\varphi(U) = \Phi'(U)$ the corresponding frequency function. Thus the counterhypotheses are $H(U) = \Phi(U + m/\sqrt{N})$, with $m > 0$. We define, for $x = H(U)$,

$$K[x] = K[H(U)] = [\Phi(U + m/\sqrt{N}) - \Phi(U)]\sqrt{N} = m\varphi(U) + O(1/\sqrt{N});$$

Further, omitting terms of lower order,

$$\frac{\delta K}{\delta x} = K'[x] = -mU, \qquad K''[x] = -\frac{m}{\varphi(U)}$$

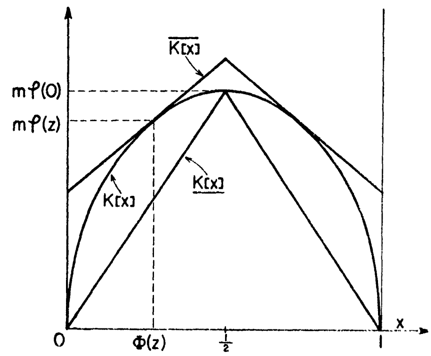

FIG. 2

We then get (cf. Fig. 2)

$$K[x] \geqq \underline{K[x]} = \begin{cases} Ax, & 0 < x \leqq \frac{1}{2}; \\ A(1-x), & \frac{1}{2} \leqq x < 1; \end{cases} A = 2m\varphi(0)$$

$$K[x] \leqq \overline{K[x]} = \begin{cases} Bx + C, & 0 < x \leqq \frac{1}{2};\ B = -mz \\ B(1-x) + C, & \frac{1}{2} \leqq x < 1;\ C = mz\Phi(z) + m\varphi(z); \end{cases}$$

where $-\infty < z \leqq 0$. Now consider the probability

$$P(a, b) = P = \lim P_N \begin{Bmatrix} [F_N^*(U) - F(U)]\sqrt{N} \leqq (a-b)F(U) + b, & -\infty < U \leqq U^*; \\ [F_N^*(U) - F(U)]\sqrt{N} \leqq (b-a)F(U) + a, & U^* \leqq U < \infty; \end{Bmatrix}$$

with $F(U^*) = \frac{1}{2}$. According to (8) we have

$$P(a, b) = \Phi(a + b) - 2e^{-2ab}\Phi(a - b) + e^{-4b(a-b)}\Phi(a - 3b).$$

If the hypotheses $H(U)$ is true, we have, omitting terms of lower order

$$P = P_H = \lim P_H \begin{Bmatrix} [H^*(U) - H(U)]\sqrt{N} \leqq (a-b)H(U) + b - K[H(U)], & -\infty < U \leqq U^*; \\ [H^*(U) - H(U)]\sqrt{N} \leqq (b-a)H(U) + a - K[H(U)], & U^* \leqq U < \infty; \end{Bmatrix}$$

with $H(U^*) = \frac{1}{2}$. Thus

$$P_{\underline{H}} = P(a - B - C, b - C) < P_H < P(a - A, b) = P_{\bar{H}}.$$

Now, $1 - P(a, b) = \alpha$ is the probability of rejecting the hypothesis F when it is true. Let $\Pr\{H, \alpha\} = 1 - P_H$ be the probability of rejecting F, when H is true. We then have, for the power function $\Pr\{H, \alpha\}$,

$$1 - P_{\bar{H}} = \underline{\Pr} < \Pr\{H, \alpha\} < \bar{\Pr} = 1 - P_{\underline{H}}.$$

On the other hand, from the assumptions concerning F and H it follows that the most powerful testing procedure should be to compute from the observations

x the normally distributed variable $M_N = \sum x_\nu / \sqrt{N}$, and reject F if $M_N < \lambda$, where $\int_{-\infty}^{\lambda} d\Phi(x) = \alpha$.

If H is true, then the probability of rejecting F will be $\Pr_{\max} = \int_{-\infty}^{\lambda+m} d\Phi(x)$. The following table shows, for some values of a and b and for $m = 1$ and $z = -2\varphi(0)$, the probability α, the limits $\underline{\Pr}$ and $\overline{\Pr}$ for the power function for the test in question, and $\Pr_{\max}$, the value of the power function for the most powerful test.

a	b	α	$\underline{\Pr}$	$\overline{\Pr}$	$\Pr_{\max}$
1.5	1.5	0.011	0.044	0.054	0.098
2.5	1.0	0.012	0.052	0.094	0.105
2.0	1.0	0.029	0.103	0.169	0.186

The values in rows 1 and 2 indicate that for the hypothesis in question, we would prefer to take $a > b$.

5. The joint distribution of the coordinates of the maximum deviation. So far, we have considered only one of the coordinates for the maximum deviation. We shall now also consider the location on the U-axis of this maximum. For sake of simplicity, we deal only with one-sided alternatives. The corresponding two-sided alternatives can be treated in the same manner.

Let $K(a, U)$ denote the probability that the maximum value of the Gaussian process $X(U)$ defined by (2) is found for $U^* \leqq U$ and that this maximum value is smaller than or equal to a. Thus, $K(a, U)$ is the joint (cumulative) distribution function for the coordinates for the maximum of $X(U)$. Further, let the corresponding frequency function be $k(a, U)$, the marginal frequency functions be $g(a)$ and $h(U)$, and the conditional ones be $g_U(a)$ and $h_a(U)$, respectively.

From (1) we have $g(a) = 4ae^{-a^2}$.

To evaluate $k(a, U)$, we calculate the probability, say P, that the process $X(U)$, before reaching the ordinate through the point U, oversteps the line $x = a$, but not the line $x = a + da$, and that $X(U)$ does not overstep the line $x = a$ after t. We then have $k(a, U) = \delta P/\delta U$.

Transforming to the $W(t)$ process, P is equal to the probability that $W(t)$, before reaching a vertical line through $t = U/(1 - U)$, oversteps the line $a(t + 1)$ but not the line $(a + da)(t + 1)$ and that $W(t)$ does not pass the line $a(t + 1)$ after t.

Using (4) and (5) we then have

$$P = \int_{-\infty}^{R} \frac{\delta}{\delta a} [1 - e^{-2a(R-Z)/t}] \frac{1}{\sqrt{2\pi t}} e^{-z^2/2t} [1 - e^{-2a(R-Z)}] \, dZ$$

where $R = a(t + 1)$. This reduces to

$$P = 4ae^{-2a^2}\Phi(a, U'), \qquad U' = (2U - 1)/\sqrt{U(1 - U)},$$

where $\Phi(x)$ is the normal (cumulative) distribution function with $m = 0$ and $\sigma = 1$. Thus

$$k(a, U) = 4ae^{-2a^2}a\varphi(aU')\, dU'/\, dU$$

where $\varphi(x)$ denotes the normal frequency function. We have further

$$h(U) = \int_0^\infty \frac{\delta P}{\delta U}\, da = \int_0^\infty k(a, U)\, da = 1, \qquad 0 < U < 1.$$

Thus

$$g_U(a) = \frac{k(a, U)}{h(U)} = 4ae^{-2a^2}a\varphi(aU')\frac{dU'}{dU}, \qquad U' = \frac{2U - 1}{\sqrt{U(1 - U)}};$$

$$h_a(U) = \frac{k(a, U)}{g(a)} = a\varphi(aU')\frac{dU'}{dU}, \qquad U' = \frac{2U - 1}{\sqrt{U(1 - U)}}.$$

Finally

$$G_U(a) = \int_0^a g_U(a)\, da = 1 - 2(a/z)e^{-(a/z)^2/2} - 2\Phi\{-a/z\}, \quad z = \sqrt{U(1 - U)},$$

$$H_a(U) = \int_0^U h_a(U) dU = \Phi(aU'), \qquad U' = (2U - 1)/\sqrt{U(1 - U)}.$$

Inspecting these conditional distributions, we see that if the value a increases, then the probability also increases that this maximum value will be in the middle of the underlying theoretical distribution $F(U)$. Further, if the expressions of Kolmogorov-Smirnov are used in their ordinary form for testing a hypothesis concerning $F(U)$, one undervalues the importance of deviations between the empirical and theoretical distributions at the extremes of $F(U)$ compared with those in the middle. An alternative procedure could then be to observe the co-ordinate U and then use the conditional distribution $G_U(a)$. In this connection, it should be kept in mind, however, that the distributions derived are limiting distributions.

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