On simplification, this also reduces to $\sum a_{ii} = \sum a_{ij}/n$, the same as obtained from the condition $B = 0$. Thus an important conclusion is reached that whenever the matrix $A = (a_{ij})$ is such that its elements satisfy the relation $\sum a_{ii} = \sum a_{ij}/n$ both the coefficients A and B of the differential equation (4) vanish simultaneously, thus leading to no solution of the problem.

Since cases II and III are excluded by our assumption $\sum a_{ii} \neq \sum a_{ij}/n$, the problem leads uniquely to the solution obtained in (9). Obviously when the matrix $A = (a_{ij})$ is either positive definite or negative definite, the relation $\sum a_{ii} \neq \sum a_{ij}/n$ is always satisfied. Thus the equality $\sum a_{ii} = \sum a_{ij}/n$ may hold only for some indefinite matrices.

COROLLARY. *Let $X_1, X_2, \cdots, X_n$ be identically distributed independent random variables with a finite second moment. If the ratio of the linear functions of random variables given by $(a_1X_1 + \cdots + a_nX_n)/(X_1 + \cdots + X_n)$ is distributed independently of the sum $X_1 + X_2 + \cdots + X_n$ then each X will follow a gamma distribution.*

PROOF. From the statement above, it follows that the conditional expectation of $(a_1X_1 + \cdots + a_nX_n)^2/(X_1 + \cdots + X_n)^2$ for the fixed sum $X_1 + \cdots + X_n$ is equal to its unconditional expectation. Here the elements of the matrix A are given by $a_{ij} = a_i a_j$ for $i, j = 1, 2, \cdots, n$ and they always satisfy the Schwartz's inequality $\sum a_i^2 > (\sum a_i)^2/n$, excluding the trivial case $\sum a_i^2 = (\sum a_i)^2/n$ which is possible when and only when all a_i's are equal, thus reducing the ratio of the linear functions to a constant. Hence the relation $\sum a_{ii} \neq \sum a_{ij}/n$ is always satisfied and the proof follows at once.

REFERENCE

[1] E. J. G. PITMAN, "The 'closest' estimates of statistical parameters," *Proc. Cambridge Philos. Soc.*, Vol. 33 (1937), pp. 212–222.

MATCHING IN PAIRED COMPARISONS

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1. One of the simplest designs for testing the effect of a treatment is the method of paired comparisons: $2n$ subjects are divided into n pairs, and within each pair the treatment is assigned at random to one of the two subjects while the other is used as a control. This method has the reputation of being most effective if the subjects within each pair are as closely matched as possible. We shall show below that while this is true in the situations occurring most commonly in practice, it is not correct universally.

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We are interested in the power of the one-sided sign test for testing the hypothesis H of no effect against the simple alternative K that the treatment has a specified positive effect.

Consider now a possible pair of subjects and assume the usual model: the score of A, B is composed of a true value a, b, an error term U, V and, in case the treatment is applied and is effective, a treatment effect t. Then if X and Y are the scores of A and B, respectively, we have under H: $X = a + U$, $Y = b + V$, while under K the quantity t is added to the score of the treated subject. We assume that U and V are identically and independently distributed according to a continuous distribution F, and denote by G the distribution of $V - U$. Then if the treatment is applied to A or B, with probability $\frac{1}{2}$ each, the probability that the score of the treated subject exceeds that of the untreated one is $\frac{1}{2}$ under H, and

$$\tfrac{1}{2}[G(t + \Delta) + G(t - \Delta)]$$

under K, where $\Delta = b - a$. Without loss of generality, Δ may be taken as nonnegative.

If A and B are perfectly matched, then $\Delta = 0$ and the probability that the treated subject has the greater score becomes $G(t)$. Perfect matching can therefore be guaranteed to give the highest power against all alternatives if and only if

$$\tfrac{1}{2}[G(t + \Delta) + G(t - \Delta)] \leqq G(t) \qquad \text{for all } t \geqq 0, \text{ all } \Delta. \tag{1}$$

This condition clearly implies that $G(u)$ is concave for $u \geqq 0$: that the converse is also true is at once obvious for $\Delta \leqq t$. For $t < \Delta \leqq 2t$, note that the values of G involved in (1) are unaltered if in the interval $[t - \Delta, \Delta - t]$ the function G is replaced by its chord. The resulting curve is concave to the right of $t - \Delta$ and (1) follows. Finally, for $\Delta > 2t$, we note that (1) is equivalent to

$$G(\Delta + t) - G(\Delta - t) \leqq G(t) - G(-t) \qquad \text{for all } t \geqq 0,\ \Delta \geqq 0. \tag{2}$$

This time replace G by its chord in the interval $[-t, t]$, to establish (1).

Matters simplify if we assume that G has a density g. Then the convexity of G is equivalent to the requirement that the symmetrical function $g(u)$ be a decreasing for $u \geqq 0$, and hence unimodal (with mode 0). In summary, a necessary and sufficient condition for perfect matching to give always the greatest power is that the density g be unimodal.

It is clear that there are distributions F of the error U for which this condition holds. The normal case is an example, since then G is again a normal distribution. However, it is also easy to give examples for which the condition is not satisfied. Let F be uniformly distributed over the union of the intervals (0, 1) and (4, 5). Then $g(u) = 0$ for $1 < |u| < 3$ and is positive for $3 < |u| < 5$. In this extreme example the gain in power may be considerable. We have $G(1) = G(3) = \frac{3}{4}$ and $G(5) = 1$. With $t = 3$ the probability that the treated subject exceeds the untreated one is $\frac{3}{4}$ when $\Delta = 0$ and $\frac{7}{8}$ when $\Delta = 2$. If we use 10 pairs and consider the treatment as significant when the response of the treated subject is higher in eight or more pairs, the significance level is .055. The power against a treatment-effect $t = 3$ is then only .526 when identical subjects are paired but rises to .880 when the subjects in each pair have a response difference $\Delta = 2$. Thus, for certain error distributions and sizes of treatment effects, it is possible to improve the power of the test substantially by purposely mismatching.[2]

It appears that to use the possibility of improving the power (when it exists), one must know the distribution G. But if G were known, one could obtain a more powerful test based on the differences themselves, instead of just on the signs of differences. This is the very common difficulty, that the choice of an optimum design depends on knowledge which a priori was assumed unavailable. However, while values of nuisance parameters, form of distributions, etc., frequently are not sufficiently well known for the validity of the test to depend on this knowledge, one does have some idea about them, which may be utilized in the design of the experiment. The statistical procedure then will be valid, whether one's ideas are correct or not. Only the sensitivity of the experiment will be affected by the accuracy of these ideas.

In the next section we shall show that g is unimodal whenever F has a unimodal density, and this is the case in most applications. However, bimodal error distributions do occur, particularly when there is the possibility of "gross error." In such cases mismatching may increase the power of the test.

2. The purpose of this section is to prove that the difference of two independent observations on a unimodal random variable has also a unimodal distribution. We note that the same is not true of the sum, as has been pointed out by Chung [1], who gives a counter example. It is also easy to see that our condition is not a necessary one by considering

$$P(X = 1) = 1/5 \text{ and } P(X = 0) = P(X = 2) = 2/5.$$

DEFINITION. We say that a random variable X is unimodal with mode m
(a) in the discrete case, if the possible values of X are equally spaced numbers $m, m \pm \Delta, m \pm 2\Delta, \cdots$, and

$$\cdots \leqq P(X = m - 2\Delta) \leqq P(X = m - \Delta) \leqq P(X = m) \geqq P(X = m + \Delta) \geqq P(X = m + 2\Delta) \geqq \cdots,$$

(b) or, in the continuous case, if X has a density function f which is increasing for $x < m$ and decreasing for $x > m$.

We shall need the following inequality.

THEOREM 1. *Let* $(a_1, a_2, \cdots, a_n)$ *be a sequence of real numbers satisfying*

$$0 \leqq a_1 \leqq \cdots \leqq a_m \geqq a_{m+1} \geqq \cdots \geqq a_n \geqq 0 \tag{3}$$

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[2] It should, however, be pointed out that the corresponding possibility does not exist if one is interested in a point estimate of the treatment effect.

for some $1 \leqq m \leqq n$. *Let* $S_k = a_1a_{1+k} + a_2a_{2+k} + \cdots + a_{n-k}a_n$ *for* $k = 0, 1, \cdots, n-1$. *Then* $S_0 \geqq S_1 \geqq \cdots \geqq S_{n-1}$.

PROOF. Fix $k \geqq 0$ and prove $S_k \geqq S_{k+1}$ for $n \geqq k+2$. For $n = k+2$ our proposition becomes

$$a_1a_{k+1} + a_2a_{k+2} \geqq a_1a_{k+2},$$

which is easily verified: $a_1a_{k+1} \geqq a_1a_{k+2}$ unless $a_{k+1} < a_{k+2}$, in which case $a_1 \leqq a_2$ and $a_2a_{k+2} \geqq a_1a_{k+2}$. We induct on n. Let there be given any sequence $(a_1, \cdots, a_n)$ satisfying (3), with $n > k+2$. We may assume $m > 1 + k$, since otherwise we have easily

$$S_k - S_{k+1} = a_1(a_{1+k} - a_{2+k}) + \cdots + a_{n-k-1}(a_{n-1} - a_n) + a_{n-k}a_n \geqq 0.$$

Since we also have

$$S_k - S_{k+1} = a_1a_{1+k} + (a_2 - a_1)a_{2+k} + \cdots + (a_{n-k} - a_{n-k-1})a_n,$$

the theorem is obvious if $m \geqq n - k$. We therefore now assume $1 + k < m < n - k$.

Let us consider the sequence $(a_1, \cdots, a_{m-1}, a_{m+1}, \cdots, a_n)$ obtained from the given sequence by dropping a_m, and let S' denote the sums of products for the new sequence. Note that the new sequence also satisfies (3). We have

$$\begin{aligned} S_k' &= (a_1a_{1+k} + \cdots + a_{m-1-k}a_{m-1}) + (a_{m-k}a_{m+1} + \cdots + a_{m-1}a_{m+k}) \\ &\quad + (a_{m+1}a_{m+1+k} + \cdots + a_{n-k}a_n) \\ &= S_k + (a_{m-k}a_{m+1} + \cdots + a_{m-1}a_{m+k}) - (a_{m-k}a_m + \cdots + a_ma_{m+k}). \end{aligned}$$

$$S_{k+1}' = S_{k+1} + (a_{m-k-1}a_{m+1} + \cdots + a_{m-1}a_{m+k+1}) - (a_{m-k-1}a_m + \cdots + a_ma_{m+k+1}).$$

When these are differenced we have, transferring the term $a_{m-k-1}a_m$,

$$\begin{aligned} S_k - S_{k+1} = (S_k' - S_{k+1}') &+ [(a_{m-k-1}a_{m+1} + \cdots + a_{m-1}a_{m+k+1}) \\ &- (a_{m-k-1}a_m + a_{m-k}a_{m+1} + \cdots + a_{m-1}a_{m+k})] \\ &+ [(a_{m-k}a_m + \cdots + a_ma_{m+k}) \\ &- (a_{m-k}a_{m+1} + \cdots + a_ma_{m+k+1})] \\ = (S_k' - S_{k+1}') &+ [a_{m-k}(a_m - a_{m+1}) + \cdots + a_m(a_{m+k} - a_{m+k+1})] \\ &- [a_{m-k-1}(a_m - a_{m+1}) + \cdots + a_{m-1}(a_{m+k} - a_{m+k+1})] \\ = (S_k' - S_{k+1}') &+ [(a_{m-k} - a_{m-k-1})(a_m - a_{m+1}) \\ &+ \cdots 2(a_m - a_{m-1})(a_{m+k} - a_{m+k+1})]. \end{aligned}$$

By the induction hypothesis, $S_k' - S_{k+1}'$ is nonnegative; by the unimodality assumption the term in square brackets is a sum of products of nonnegative terms. We conclude $S_k \geqq S_{k+1}$.

We can now establish the desired result.

THEOREM 2. *If X and Y are independent observations on the same unimodal random variable, then $X - Y$ is unimodal.*

We prove the theorem in three parts.

PART I. If X has as possible values only finitely many integers, the theorem is an immediate consequence of the preceding one. The a's are taken to be the probabilities of the successive possible values of X. Since $P(X - Y = k) = S_k$ for k a positive integer, and since $X - Y$ has a distribution symmetric about 0, the theorem follows.

PART II. Let the possible values of X now be numbers of the form $r\Delta$, where $\Delta > 0$ and r is any integer. For simplicity we may assume 0 to be a mode. For every positive integer s, define

$$X'_s = \begin{cases} X \text{ if } |X| \leqq s, \\ 0 \text{ if } |X| > s, \end{cases} \qquad Y'_s = \begin{cases} Y \text{ if } |Y| \leqq s \\ 0 \text{ if } |Y| > s. \end{cases}$$

That $X'_s - Y'_s$ has a unimodal distribution is an immediate consequence of Part I. But since $P(X'_s - Y'_s \neq X - Y) \rightarrow 0$ as $s \rightarrow \infty$, we see that $X - Y$ must also have a unimodal distribution.

PART III. Now suppose X has a density f, with mode at m. For each positive integer s, define

$$X''_s = [(X - m)\sqrt{s}]/\sqrt{s},$$

where $[u]$ denotes the greatest integer less than u. The cumulative distribution G''_s of $X''_s - Y''_s$ cannot ever differ from G by more than a quantity which tends to 0 as $s \rightarrow \infty$. However, G''_s is unimodal, by Part II. If G were not unimodal, we could find $\epsilon > 0$, $\Delta > 0$, and $u - \Delta > 0$ such that $G(u - \Delta) + G(u + \Delta) + \epsilon < 2G(u)$, which would yield a contradiction.

REFERENCE

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NOTE ON A THEOREM OF LIONEL WEISS[1]

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1. Introduction. In a recent paper [1] it was pointed out by Lionel Weiss that the class of sequential probability ratio tests is complete in a very strong sense. The purpose of the present note is to show how this result can be derived from a

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