# RANDOM FUNCTIONS SATISFYING CERTAIN LINEAR RELATIONS, II[1].

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**0. Introduction and summary.** We consider here another aspect of a problem mentioned in a previous paper [2]. We shall be concerned with one-dimensional, real-valued random functions (r.f.) $X(t)$, defined for all $t$ and such that any sample taken at equidistant $t$-points satisfies a linear relation which is an analogue of one or other of the stochastic difference relations which are used for the analysis of discrete-parameter time-series. More specifically, we assume that there exist $k$ continuous and real-valued functions $\alpha_1(h), \cdots, \alpha_k(h)$ of $h \geqq 0$ such that for any $h > 0$ and any $t$, the sequence

$$(1) \qquad \{X(t + [n + k]h) + \alpha_1(h)X(t + [n + k - 1]h) + \cdots + \alpha_k(h)X(t + nh)\}, \quad n = 0, \pm 1, \cdots,$$

satisfies certain conditions about independence or noncorrelation. In Section 1, we consider hypotheses concerning correlation, and find that the functions $\alpha_j(h)$ are restricted to certain forms. We also find that the assumption of zero serial correlations in the sequence (1) for all $h > 0$ implies that $X(t)$ is deterministic. In Section 2, we consider hypotheses concerning independence, and find the functions $\alpha_j(h)$ to be restricted as before.

**1. Hypotheses about correlation.** In this section, we assume $X(t)$ to have a finite variance and to be continuous in mean square for all $t$. We shall consider first processes with stationary covariance, and then all processes with continuous covariance. Although the former case is included in the latter, we find that the restriction of stationarity enables us to consider somewhat less restrictive hypotheses. By analogy with the definition of "$m$-dependence" [3], we shall use the following

Definition. If $\{X_n\}$ has a finite variance for all $n$, and $X_n$ and $X_{n'}$ are noncorrelated for $|n - n'| > m$, the sequence $\{X_n\}$ is said to be *$m$-correlated.*

Theorem 1. *Let $X(t)$ be a random function with a continuous, stationary covariance, and let there exist a positive integer $m$ and a set of $k$ real-valued, continuous functions $\alpha'_1(h), \alpha'_2(h), \cdots, \alpha'_k(h)$ of $h \geqq 0$, such that*

(i) *for any $h > 0$, the random variables*

$$(2) \qquad Y(\alpha'; h; n) = X([n + k]h) + \alpha'_1(h)X([n + k - 1]h) + \cdots + \alpha'_k(h)X(nh), \quad n = 0, \pm 1, \cdots,$$

*form an $m$-correlated sequence;*

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(ii) *there is no $h > 0$ for which such a relation holds for some $m$ with less than $k$ coefficients, say $\beta_1(h), \beta_2(h), \cdots, \beta_l(h)$, with $l < k$, instead of the $\alpha_j'(h)$.*

*Then there exists a uniquely determined set of $k$ continuous, real-valued functions $\alpha_1(h), \cdots, \alpha_k(h)$ with the following properties:*

(a) *For any $h > 0$, the sequence $\{Y(\alpha; h; n)\}$,* for $n = 0, \pm 1, \cdots$ *is $(k-1)$-correlated, that is to say $m = k - 1$. It is $(k-2)$-correlated if and only if $X(t)$ is deterministic and satisfies the relation*

$$X(t + kh) + \alpha_1(h)X(t + [k-1]h) + \cdots + \alpha_k(h)X(t) = 0 \tag{3}$$

*with probability 1 for any $h > 0$ and any $t$.*

(b) *Every root $\gamma_j(h)$ of the equation in $x$,*

$$x^k + \alpha_1(h)x^{k-1} + \cdots + \alpha_k(h) = 0, \tag{4}$$

*can be written in the exponential form* $\exp(\lambda_j h)$, *where the $\lambda_j$ are independent of $h$ and are roots, of respective multiplicities $k_j$, of an algebraic equation of degree $k$ with real coefficients which are independent of $h$. Further, the real parts of the $\lambda_j$ are nonpositive, and $k_j = 1$ if $\lambda_j$ is purely imaginary. The root* $\exp(\lambda_j h)$ *of* (4) *has multiplicity $k_j$.*

Proof. It is no restriction to assume $E\{X(t)\} = 0$ and $E\{X^2(t)\} = 1$. We shall also write $\rho(h)$ for $E\{X(t)X(t+h)\}$ and define the operator $U_h$ by $U_h f(t) = f(t+h)$.

Taking $\alpha_0'(h) = 1$, and using the $m$-correlation property of the sequence (2), we have for any $h > 0$ and for any integer $n > \max(m-k, 0)$

$$\left\{\sum_{j=0}^{k} \alpha_j'(h) U_h^{k-j}\right\} \left\{\sum_{j=0}^{k} \alpha_j'(h) U_h^{j}\right\} \rho(nh) = 0. \tag{5}$$

Thus for any $h > 0$, we have a linear difference equation satisfied by the sequence $\rho(nh)$. Hence, $\rho(nh)$ can be expressed as a function of $n$ and the roots of the characteristic equation

$$\left\{\sum_{j=0}^{k} \alpha_j'(h) x^{k-j}\right\} \left\{\sum_{j=0}^{k} \alpha_j'(h) x^{j}\right\} = 0. \tag{6}$$

For the present, let us consider some fixed $h > 0$, and let $\gamma_j(h)$ for $j = 1, 2, \cdots, p, p+1, \cdots, p+p'$, be all the distinct roots, of moduli not exceeding 1, of equation (6); let $k_j$ be the multiplicity of the root $\gamma_j(h)$, and let

$$|\gamma_j(h)| \begin{cases} = 1 & \text{for } j = 1, 2, \cdots, p, \\ < 1 & \text{for } j = p+1, p+2, \cdots, p+p'. \end{cases} \tag{7}$$

Since the reciprocal of any root of (6) is also a root, we see that the whole set of distinct roots consists of

$$\gamma_j(h), \quad j = 1, 2, \cdots, p+p', \qquad 1/\gamma_j(h), \quad j = p+1, \cdots, p+p',$$

each root with suffix $j$ being of multiplicity $k_j$. Hence, for all integers $n > \max(m - k, 0)$, we get from (5)

$$\rho(nh) = \sum_{j=1}^{p} \gamma_j^n(h) \left\{ \sum_{s=0}^{k_j-1} a_{js}(h) n^s \right\} + \sum_{j=p+1}^{p+p'} \gamma_j^n(h) \left\{ \sum_{s=0}^{k_j-1} b_{js}(h) n^s \right\} + \sum_{j=p+1}^{p+p'} \gamma_j^{-n}(h) \left\{ \sum_{s=0}^{k_j-1} c_{js}(h) n^s \right\}, \qquad n > \max(m - k, 0), \tag{8}$$

where the $a$'s, $b$'s and $c$'s are constant with respect to $n$, but continuous functions of $h$. Since $|\rho(nh)| \leqq 1$, the last term in (8) drops out entirely and there can be no positive powers of $n$ in the first term on the right side of (8), so that

$$\rho(nh) = \sum_{j=1}^{p} \gamma_j^n(h) a_{j0}(h) + \sum_{j=p+1}^{p+p'} \gamma_j^n \left\{ \sum_{s=0}^{k_j-1} b_{js}(h) n^s \right\}, \quad n > \max(m - k, 0). \tag{9}$$

Therefore, the sequence $\{\rho(nh)\}$ satisfies a homogeneous, linear difference equation (with right member zero) of order $k' = p + \sum_{j=1}^{p'} k_{p+j}$. Now, the roots of (6) comprise the roots of

$$x^k + \alpha_1'(h) x^{k-1} + \cdots + \alpha_k'(h) = 0 \tag{10}$$

and their reciprocals. Hence, it follows that every root of (6) of unit modulus is of even multiplicity. Consequently, it is easy to see that $k' \not> k$. On the other hand, if $k' < k$, we can use the fact that the sequence $\{\rho(nh)\}$ satisfies a linear relation of order $k'$ to show that condition (ii) of the theorem is violated. Thus $k' = k$, and for every $h > 0$ there exist $k$ real numbers $\alpha_1(h), \cdots, \alpha_k(h)$ such that.

$$\sum_{j=0}^{k} \alpha_j(h) \rho([n - j]h) = 0 \quad \text{any } n > m_0 = \max(m, k) \tag{11}$$

This set of $k$ numbers is unique for every $h > 0$, since if there were two non-identical relations of order $k$ of the type of (11), we could derive from them a relation of lower order, and this would contradict assumption (ii) of the theorem. The equation

$$x^k + \alpha_1(h) x^{k-1} + \cdots + \alpha_k(h) = 0 \tag{12}$$

has $p$ distinct, nonrepeated roots $\gamma_j(h)$, where $j = 1, 2, \cdots, p$, of unit modulus, and $p'$ distinct roots $\gamma_j(h)$, where $j = p + 1, \cdots, p + p'$, of moduli less than 1 and respective multiplicities $k_j$.

From (11) we know that for any $h > 0$ and any integer $r > 0$, the sequence $\rho(nh/r)$, for $n = m_0 + 1, \ m_0 + 2, \cdots$, satisfies the relation

$$\sum_{j=0}^{k} \alpha_j(h/r) \rho\{(n - j)h/r\} = 0. \tag{13}$$

From (13) we shall now derive a relation satisfied by the sequence $\rho(nh)$. For this purpose, let $\alpha_1(r; h/r), \alpha_2(r; h/r), \cdots, \alpha_k(r; h/r)$ be the set of numbers determined uniquely by the property that the roots of

$$x^k + \alpha_1(r; h/r) x^{k-1} + \cdots + \alpha_k(r; h/r) = 0 \tag{14}$$

are the $r$th powers $\gamma_j^r(h/r)$ of the characteristic roots of (13). Let $\beta_i$, for $i = 0, 1, 2, \cdots$, be defined by the identity

$$(15) \qquad \sum_{i=0}^{rk-k} \beta_i x^{rk-k-i} \equiv \frac{x^{kr} + \alpha_1(r; h/r)x^{(k-1)r} + \cdots + \alpha_k(r; h/r)}{x^k + \alpha_1(h/r)x^{k-1} + \cdots + \alpha_k(h/r)}.$$

Then it is easily verified that

$$(16) \quad \sum_{i=0}^{rk-k} \beta_i \sum_{j=0}^{k} \alpha_j(h/r)\rho\{(nr - i - j)h/r\} = \sum_{j=0}^{k} \alpha_j(r; h/r)\rho\{(n - j)rh/r\}.$$

Consequently,

$$(17) \qquad \sum_{j=0}^{k} \alpha_j(r; h/r)\rho\{(n - j)h\} = 0 \qquad n > m_0.$$

But the uniqueness of (11) gives us $\alpha_j(r; h/r) = \alpha_j(h)$ and hence $\gamma_j^r(h/r) = \gamma_j(h)$. Then, by a standard argument using continuity, we have $\gamma_j(h) = \exp(\lambda_j h)$ for all $h \geqq 0$, where $\lambda_j$ is some constant. We have thus proved (b).

Next, from (9) by equating the expressions for $\rho(nh)$ and $\rho(h')$, so that $h' = nh$, we get

$$a_{j0}(h) = a_{j0}(nh), \qquad b_{js}(h)n^s = b_{js}(nh).$$

This gives us $a_{j0}(h) = a_{j0}$ and $b_{js}(h) = b_{js}h^s$, where the $a_{j0}$ and $b_{js}$ are constants. Hence,

$$(18) \qquad \rho(h) = \sum_{j=1}^{p'} a_{j0} \exp(\lambda_j h) + \sum_{j=p+1}^{p+p'} \sum_{s=0}^{k_j-1} b_{js} h^s \exp(\lambda_j h), \qquad h > 0.$$

From this form of $\rho(h)$, we immediately have

$$(19) \qquad E\{Y(\alpha; h; 0)X(t)\} = 0 \qquad h \geqq 0, \quad t \leqq 0.$$

It follows trivially that the $Y$-sequence is $(k - 1)$-correlated. Thus the first part of (a) is proved.

Finally, suppose that the $Y$-sequence is $(k - r)$-correlated for some $r \geqq 2$. This together with (19) implies that

$$(20) \qquad E\{Y(\alpha; h; j)X(rh)\} = 0 \qquad j = 1, 2, \cdots.$$

Also, since $(k - r)$-correlation with $r \geqq 2$ implies $(k - 2)$-correlation, we have

$$(21) \qquad E\{Y(\alpha; rh; 0)X(rh)\} = 0.$$

Now, by the procedure used in (14) to (16), we can obtain the relation

$$Y(\alpha; rh; 0) = \beta_0 Y(\alpha; h; rk - k) + \cdots + \beta_{rk-k} Y(\alpha; h; 0),$$

which leads to

$$(22) \qquad E\{Y(\alpha; h; 0)X(rh)\} = 0.$$

This implies that the $Y$-sequence is $(k - r - 1)$-correlated. Hence, by induction we can show that the assumption of $(k - 2)$-correlation in the $Y$-sequence implies

$$(23) \qquad E\{Y^2(\alpha; h; 0)\} = 0,$$

which proves the second part of (a).

REMARKS. The uniqueness of the coefficients $\alpha_j(h)$ is a consequence of the restriction $|\gamma_j(h)| \leqq 1$. There is another unique set of $k$ coefficients under the restriction $|\gamma_j(h)| \geqq 1$; in this case the roots are reciprocals of those in the previous. This is merely a consequence of the fact that, on account of stationarity, the covariance function of $Y(t) = X(-t)$ is the same as that of $X(t)$; the reversal of the $t$-axis transforms the linear function with characteristic roots $\gamma_j(h)$ into one whose roots are $1/\gamma_j(h)$.

This result may be compared with some of the results of Doob [1]. The first $k - 1$ derivatives in mean square of a stationary r.f. exist if and only if $\rho(h)$ has a derivative of order $2(k - 1)$ at the origin, and this condition is completely equivalent to the finiteness of the moment of order $2(k - 1)$ of the spectral function. From these facts and Theorems 3.10, 4.8 and 4.9 of Doob [1], we can derive the

COROLLARY. *A one-dimensional, stationary Gaussian process is a component of a $k$-dimensional* "t.h.G.M." *process if and only if it is a Gaussian process satisfying the assumptions of Theorem* 1. *Furthermore, it is a* "t.h.G.M$_k$" *process if and only if, in addition, its $(k - 1)$th derivative in mean square exists.*

We shall now state a result similar to Theorem 1 which holds without the restriction of stationarity.

THEOREM 2. *Let $X(t)$ be continuous in mean square for all $t$, with $E\{X(t)\} = 0$. Let there exist a positive integer $m$, and $k$ continuous functions, $\alpha_1(h), \cdots, \alpha_k(h)$, of $h \geqq 0$, such that*

$$E\{[X(t + kh) + \alpha_1(h)X(t + [k - 1]h) + \cdots + \alpha_k(h)X(t)]X(t - nh)\} = 0, \quad n = m, \; m + 1, \cdots, \tag{24}$$

*assuming that such a relation does not hold with any other set of $\alpha$'s for any $h$ (and any $m$).*

*Then the roots of* (4) *can be written in the form* $\exp(\lambda_j h)$, *where the $\lambda_j$ are the roots of an algebraic equation of degree $k$ with real coefficients. Furthermore,* (24) *is true for all integers $n \geqq 0$, and if it also holds for $n = -1$, then it holds for all $n$, and Equation* (3) *is true with probability* 1.

The proof is similar to that of Theorem 1, and is thus omitted.

**2. Hypotheses of independence.** Finally, we shall consider a result of this type, but with noncorrelation replaced by independence. We shall see that the $Y$-sequence is necessarily $(k - 1)$-dependent, but we are unable to determine whether it can be $(k - 2)$-dependent without triviality.

THEOREM 3. *Let $X(t)$ be a real-valued* r.f., *continuous in probability for all $t$. Let there exist a positive integer $m$, and $k$ real-valued, continuous functions $\alpha_1(h), \cdots, \alpha_k(h)$, of $h \geqq 0$ such that*

(i) *for any $t$ and any $h > 0$, the random variables*

$$Y(t, h; n) = X(t + [n + k]h) + \alpha_1(h)X(t + [n + k - 1]h) + \cdots + \alpha_k(h)X(t + nh), \qquad n = 0, \pm 1, \cdots,$$

*form an $m$-dependent sequence;*

(ii) *there is no other set of* $\alpha$'s *for which this is true for some* $m$; *and*

(iii) $\alpha_k(h) \neq 0$ *for any* $h$.

*Then the roots of* (4) *can be written in the form* $\exp(\lambda_j h)$, *where the* $\lambda_j$ *are roots of an algebraic equation of degree* $k$ *with real coefficients. Furthermore,* $m = k - 1$.

PROOF. As before, we take any positive $h$ and any positive integer $r > m - k$, and let $h' = h/r$. On account of $m$-dependence, we know that, for any integer $l > m$ and any positive integers $p$ and $q$, the set of random variables $\{Y(t, h'; n)\}$ for $n = 0, 1, \cdots, (p + k)r$ is independent of the set $\{Y(t, h'; -l - n)\}$ for $n = 0, 1, \cdots, (q + k)r$. Now let

$$S_n(r) = \sum_{j=0}^{kr-k} \beta_j Y(t, h';\ nr + kr - k - j), \qquad n = 0, 1, \cdots, p. \tag{25}$$

$$\begin{aligned} T_n(r, l) &= \sum_{j=0}^{kr-k} \beta_j Y(t, h';\ -nr - l - j) \\ &= \sum_{j=0}^{kr-k} \beta_j Y[t + (k - l)h', h';\ -nr - k - j], \end{aligned} \qquad j = 0, 1, \cdots, q, \tag{26}$$

where the $\beta_j$ are as in (15). Then the random variables $S_n(r)$ for $n = 0, 1, \cdots, p$ are independent of $T_n(r, l)$ for $n = 0, 1, \cdots, q$.

By the method used in (14) to (16), we shall now derive a new linear difference function which generates a sequence having the properties of $\{Y(t, h;\ n)\}$. For this purpose, let

$$\begin{aligned} Y'(t, h, n) &= X[t + (n + k)h] \\ &\quad + \alpha_1(r; h')X[t + (n + k - 1)h] + \cdots + \alpha_k(r; h')X[t + nh]. \end{aligned} \tag{27}$$

Then it is easily verified that

$$S_n(r) = Y'(t, h;\ n), \qquad T_n(r, l) = Y'[t + (k - l)h', h;\ -n - k]. \tag{28}$$

For $l = r + k$, we have from the second expression that

$$\begin{aligned} T_n(r, r + k) &= Y'(t - rh', h; -n - k) \\ &= Y'(t, h; -n - k - 1). \end{aligned} \tag{29}$$

Since the $S_n(r)$ for $n = 0, 1, \cdots, p$ are independent of $T_n(r, r + k)$ for $n = 0, 1, \cdots, q$, it is clear that the $Y'$-sequence is $k$-dependent. Hence, by assumption (ii) of the theorem

$$Y'(t, h; n) = Y(t, h; n), \tag{30}$$

so that $\alpha_j(r; h/r) = \alpha_j(h)$. This leads as before to $\gamma_j(h) = \exp(\lambda_j h)$.

Finally, from (28) and (30) we know that for any $l > m$ and any positive integers $p$ and $q$, the random variables $Y(t, h; n)$ for $n = 0, 1, \cdots, p$ are independent of $Y[t + (k - l)h/r,\ h;\ -n - k]$ for $n = 0, 1, \cdots, q$. With $l = m + 1$, as $r \to \infty$ the second set of random variables converges in probability to $Y[t, h; -n - k]$ for $n = 0, 1, \cdots, q$ and therefore is independent of $Y(r, h; n)$ for $n = 0, 1, \cdots, p$. In other words, the $Y$-sequence is $(k - 1)$-dependent, which is to say that $m = k - 1$.

COROLLARY. *Any random function $X(t)$ having the property that for any $t$ and any $h > 0$ the increments $[X(t + h) - X(t)]$, $[X(t + 2h) - X(t + h)]$, $\cdots$ form an $m$-dependent sequence, where $m$ is some nonnegative integer, is a random function with independent increments ("additive process").*

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