SEQUENTIAL LIFE TESTS IN THE EXPONENTIAL CASE[1]

BY BENJAMIN EPSTEIN AND MILTON SOBEL

Wayne University and *Cornell University*

1. Introduction and summary. This paper describes sequential life test procedures, considering, as in a recent paper [4] devoted to nonsequential methods, the special case in which the underlying distribution of the length of life is given by the exponential density

$$f(x, \theta) = e^{-x/\theta}/\theta, \qquad x > 0. \tag{1}$$

The unknown parameter $\theta > 0$ can be thought of physically as the mean life.

Our primary aim is to test the simple hypothesis $H_0 : \theta = \theta_0$ against the simple alternative $H_1 : \theta = \theta_1$, where $\theta_1 < \theta_0$, with type I and II errors equal to preassigned values α and β, respectively. The test is carried out by drawing n items at random from the population and placing them all on a life test. We consider both the replacement case, in which failed items are immediately replaced by new items, and the nonreplacement case.

The test can be terminated either at failure times with rejection of H_0, or at any time between failures with acceptance of H_0. Since abnormally long intervals between failures furnish "information" in favor of H_0 and abnormally short intervals furnish "information" in favor of H_1, these features are not only reasonable but actually desirable. Similar problems involving a continuous time parameter have recently appeared [3], [5].

In this paper we obtain likelihood ratio tests and give approximate formulae for the O.C. (operating characteristic) curve, for the expected number of failures $E_\theta(r)$, and for the expected waiting time $E_\theta(t)$ before a decision is reached. In the replacement case where the number of items on test throughout the experiment is the same, namely n, it is shown that $E_\theta(t) = (\theta/n)E_\theta(r)$. A table giving approximate values of $E_\theta(r)$ for certain choices of θ_0/θ_1, α, and β is given for the replacement case. Some calculations of exact $L(\theta)$ and $E_\theta(r)$ values using formulae in [1] and [3] are reported. Several numerical examples are worked out.

2. Basic formulae. Wald's work on sequential analysis [8] can be used virtually without modification in a situation where decisions are made continuously. In fact, in a truly continuous situation, Wald's formulae become exact, since there is then no excess over the boundary. It will become clear as we proceed that, in the problem at hand, the situation can be termed semicontinuous (not to be confused with the concept of the same name in real variable theory). There is no excess over the boundary used for accepting H_0, but in general there will be some excess over the boundary used in accepting H_1.

Let us assume that from the underlying exponential p.d.f. (1), n items are

Received March 8, 1954.

[1] Research sponsored in part by the Office of Naval Research and the Office of Ordnance Research of the U. S. Army.

drawn at random and placed on life test. We wish to test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ with type I error $= \alpha$ and type II error $= \beta$. Since information is available continuously, a continuous analogue of the sequential probability ratio test of Wald can be used. The decision as time unfolds depends on

$$B < (\theta_0/\theta_1)^r \exp[-(1/\theta_1 - 1/\theta_0)V(t)] < A \tag{2}$$

where B and A are constants, depending on α and β, such that $B < 1 < A$. The decision to continue experimentation is made as long as the inequality (2) holds. At the time the experiment is stopped, if the first inequality in (2) is violated, we accept H_0; if the second inequality is violated, we accept H_1. As in Wald's case, the test obtained by setting $B = \beta/(1 - \alpha)$ and $A = (1 - \beta)/\alpha$ is a satisfactory solution of the problem from a practical point of view. Details are given in remarks 1 and 2 in Section 4.

In (2), $V(t)$ is a statistic which can be interpreted as the total life observed up to time t. In the replacement case

$$V(t) = nt, \tag{3}$$

while in the nonreplacement case[2]

$$\begin{aligned} V(t) &= \sum_{i=1}^{r} (n - i + 1)(x_i - x_{i-1}) + (n - r)(t - x_r) \\ &\equiv \sum_{i=1}^{r} x_i + (n - r)(t - x_r), \end{aligned} \tag{4}$$

where x_i denotes the time of the ith failure, with $x_0 = 0$.

To graph the data continuously in time, it is convenient to write (2) in the form

$$-h_1 + rs < V(t) < h_0 + rs, \tag{5}$$

where h_0, h_1, and s are positive constants given by

$$h_0 = \frac{-\log B}{1/\theta_1 - 1/\theta_0}, \qquad h_1 = \frac{\log A}{1/\theta_1 - 1/\theta_0}, \qquad s = \frac{\log(\theta_0/\theta_1)}{1/\theta_1 - 1/\theta_0}. \tag{6}$$

Further, it can be shown ([8], pp. 48–50) that the O.C. curve, that is, the probability of accepting H_0 when θ is the true parameter value, is given approximately by a pair of parametric equations

$$L(\theta) = \frac{A^h - 1}{A^h - B^h}, \qquad \theta = \frac{(\theta_0/\theta_1)^h - 1}{h(1/\theta_1 - 1/\theta_0)}, \tag{7}$$

by letting the parameter h run through all real values.

[2] In the nonreplacement case it may happen that no decision has been reached by the time $t = x_n$, when all n items have failed. This will then require that we either put more items on test and wait until (2) is violated or else have a rule which will tell us how to terminate the experiment and with what decision at $t = x_n$. Fortunately n is often at our disposal and so can be chosen sufficiently large so that the probability of reaching no decision by time x_n is negligible. For large enough n, it really makes very little difference how we truncate experimentation. We could, for example, adopt the rule that H_1 is accepted if (2) is satisfied for all $t \leqq x_n$.

The values of $L(\theta)$ at the five points $\theta = 0, \theta_1, s, \theta_0$, and ∞ enable one to sketch the entire curve. These values are respectively 0, β, $\log A/(\log A - \log B)$, $1 - \alpha$, and 1.

We now give, in terms of $L(\theta)$, an approximate formula for $E_\theta(r)$, the expected number of observations required to reach a decision when θ is the true parameter value. Since the logarithm of the middle expression in (2) is either $\log B$ or $\log A$ at the time experimentation stops, we have, neglecting only the excess over $\log A$,

$$(8)\quad E_\theta(r) \log (\theta_0/\theta_1) - E_\theta(V(t))[1/\theta_1 - 1/\theta_2] \sim L(\theta) \log B + [1 - L(\theta)] \log A.$$

It is proved in the next section that

$$(9)\qquad E_\theta(V(t)) = \theta E_\theta(r).$$

Hence we have from (8) and (9)

$$(10)\quad E_\theta(r) \sim \begin{cases} \dfrac{L(\theta) \log B + [1 - L(\theta)] \log A}{\log (\theta_0/\theta_1) - \theta(1/\theta_1 - 1/\theta_0)} = \dfrac{h_1 - L(\theta)(h_0 + h_1)}{s - \theta}, & \theta \neq s \\[2ex] \dfrac{-\log A \quad \log B}{[\log (\theta_0/\theta_1)]^2} = \dfrac{h_0 h_1}{s^2}, & \theta = s. \end{cases}$$

If we let $k = \theta_0/\theta_1$, the approximate values of $E_\theta(r)$ become particularly simple when $\theta = \theta_1, s$, or θ_0. They are

$$(11)\quad \begin{aligned} E_{\theta_1}(r) &\sim [\beta \log B + (1 - \beta) \log A] \,/\, [\log k - (k - 1)/k], \\ E_s(r) &\sim -\log A \quad \log B \,/\, (\log k)^2, \\ E_{\theta_0}(r) &\sim [(1 - \alpha) \log B + \alpha \log A] \,/\, [\log k - (k - 1)]. \end{aligned}$$

In Table 1 we give $E_\theta(r)$ for five values of θ $(0, \theta_1, s, \theta_0, \infty)$, for four values of k $(3/2, 2, 5/2, 3)$, and for the four number pairs (α, β) which can be made with the numbers .01 and .05.

3. A basic identity. In this section (9) is derived. While this result can be obtained as a consequence of a theorem of Doob on continuous parameter martingales ([2], p. 376), a simpler proof is desirable. We shall consider the replacement case, where $V(t) = nt$, although the proof can be trivially modified so as to hold in nonreplacement and truncated situations.

In the replacement case (9) becomes

$$(12)\qquad E_\theta(t) = E_\theta(r)\, \theta/n.$$

Thus we are relating expected waiting time to reach a decision to the expected number of failures.

To prove (12) we introduce a "large" integer N and let x_N denote the time of the Nth failure. Let t denote the (first) time at which the inequality (2) is violated or the time x_N, whichever comes sooner. Then we can write

$$(13)\qquad x_N = t + (x_{r+1} - t) + (x_{r+2} - x_{r+1}) + \cdots + (x_N - x_{N-1}).$$

TABLE 1

Approximate values of $E_\theta(r)$ for sequential tests for various values of $k = \theta_0/\theta_1$ and α, β

$k = \theta_0/\theta_1$		3/2		2		5/2		3	
α		.01	.05	.01	.05	.01	.05	.01	.05
θ	β								
0	.01	11	7	7	4	5	3	4	3
	.05	11	7	7	4	5	3	4	3
θ_1	.01	62.4	40.3	23.3	15.1	14.2	9.20	10.4	6.74
	.05	60.4	36.7	22.6	13.7	13.8	8.38	10.1	6.14
s	.01	128	82.7	43.9	28.3	25.1	16.2	17.5	11.3
	.05	82.7	52.7	28.3	18.0	16.2	10.3	11.3	7.18
θ_0	.01	47.6	44.2	14.7	13.6	7.71	7.16	5.00	4.63
	.05	30.8	28.0	9.48	8.64	4.99	4.54	3.23	2.94
∞	any	0	0	0	0	0	0	0	0

Three actions are possible: accept H_1 before x_N, in which case $t = x_r$; accept H_0 before x_N, in which case $x_r < t < x_{r+1}$; or take no action before x_N, in which case $t = x_N$ and $r = N$. Since N is fixed in advance

$$E(x_N) = N\theta/n. \tag{14}$$

Further, it is easily verified that the $(N - r)$ random variables $(x_{r+1} - t)$, $(x_{r+2} - x_{r+1}), \cdots, (x_N - x_{N-1})$ are independently and identically distributed with the exponential density $(n/\theta)e^{-nx/\theta}$, for $x > 0$. Hence if we take the expectation of both sides of (14), first holding r fixed and then taking the expectation with respect to r, we obtain

$$N\theta/n = E_\theta(t; N) + [N - E_\theta(r; N)]\theta/n \tag{15}$$

or

$$E_\theta(t; N) = E_\theta(r; N)\theta/n. \tag{16}$$

Formula (16) holds for all N. As $N \to \infty$ the probability of coming to a decision before x_N tends to unity. Moreover as $N \to \infty$

$$E_\theta(r; N) \uparrow E_\theta(r), \qquad E_\theta(t; N) \uparrow E_\theta(t), \tag{17}$$

where $E_\theta(r)$ and $E_\theta(t)$ are respectively the expected number of failures and expected waiting time to reach a decision if $N = \infty$. Thus it follows, letting $N \to \infty$, that (16) becomes (12). The nonreplacement case can be treated in exactly the same way, because

$$V(x_n) = V(t) + (n - r)(x_{r+1} - t) + (n - r - 1)(x_{r+2} - x_{r+1}) + \cdots + (x_n - x_{n-1}). \tag{18}$$

As before, $t = x_r$, if H_1 is accepted; $x_r < t < x_{r+1}$, if H_0 is accepted; and $t = x_n$, if no decision is reached by the time all n items have failed. The last $(n - r)$ components on the right side of (18) are mutually independent random variables, each distributed with the p.d.f. (1). Thus it follows as in the replacement case that

$$E_\theta(V(t); n) = \theta E_\theta(r; n). \tag{19}$$

As n increases, $E_\theta(r; n) \uparrow E_\theta(r)$, the expected number of failures in reaching a decision in the replacement case. Thus no matter how we decide to terminate experimentation, $E_\theta(V(t); n)$ can be replaced by $E_\theta(V(t)) = \theta E_\theta(r)$, when n is large. In practice, for "large" n one could take $n > 3 \max_\theta E_\theta(r)$.

While (12) relates expected waiting time to the expected number of failures in the replacement case, (19) relates expected total life (not waiting time) to the expected number of failures in the nonreplacement case. Actually one has to know the probability distribution of r in order to compute $E_\theta(t)$ exactly in the nonreplacement case. It can be shown that in the nonreplacement case the formula for $E_\theta(t)$ is given by

$$E_\theta(t) = \sum_{k=1}^{n} \Pr(r = k \mid \theta) E_\theta(X_{k,n}), \qquad E_\theta(X_{k,n}) = \theta \sum_{i=1}^{k} \frac{1}{n - i + 1}. \tag{20}$$

In the replacement case one has, analogous to (20),

$$E_\theta(t) = \sum_{k=1}^{\infty} \Pr(r = k \mid \theta) E_\theta(X_{k,n}), \qquad E_\theta(X_{k,n}) = \frac{k\theta}{n}. \tag{20'}$$

where n is the sample size maintained throughout the experiment. Thus, in the replacement case (20′) clearly becomes (12). Equation (20) is valid for all life test procedures which involve nonreplacement. Similarly (20′) holds for all life test procedures, where items which fail are replaced.[3]

4. Some remarks. This section contains three extended remarks on certain aspects of Sections 2 and 3.

Remark 1. Upper and lower bounds for $L(\theta)$ and $E_\theta(r)$. The formulae for $L(\theta)$ and $E_\theta(r)$, given by (7) and (10) respectively, are approximations to the actual $L(\theta)$ and actual $E_\theta(r)$ arising from the use of the semi-continuous sequential decision rule specified by the inequalities (2). The question arises as to how good these approximations are. A modification of the results of Wald on bounds for the

[3] Here, "all" means truncated or untruncated, sequential, or any similar procedures. Of course the probability distribution of r does depend on the procedure which is followed. In [6] explicit formulae for $\Pr(r = k \mid \theta)$ are worked out for three procedures.

O.C. and ASN curves in the binomial case [8] and of results of Herbach [7] on the discrete Poisson yields the following bounds on the actual $L(\theta)$ and $E_\theta(r)$:

$$\frac{A^h - 1}{A^h - B^h} \leqq L(\theta) \leqq \frac{(kA)^h - 1}{(kA)^h - B^h}, \qquad h \neq 0 \text{ (that is, for } \theta \neq s), \tag{21}$$

$$\frac{L(\theta) \log B + [1 - L(\theta)] \log A}{\log k - \theta(1/\theta_1 - 1/\theta_0)} \begin{Bmatrix} \leqq \\ \geqq \end{Bmatrix} E_\theta(r) \begin{Bmatrix} \leqq \\ \geqq \end{Bmatrix} \frac{L(\theta) \log B + [1 - L(\theta)][\log A + \log k]}{\log k - \theta(1/\theta_1 - 1/\theta_0)} \tag{22}$$

where the upper inequality signs hold for $\theta < s$ and the lower inequality signs hold for $\theta > s$.

One unpleasant feature of the bounds given in (22) is that they involve $L(\theta)$, which is unknown. However, this matters little in actual practice because the limits on $L(\theta)$ given by (21) are quite close together for the range of values of k and (α, β) covered in Table 1. Thus, for example, for

$$k = \theta_0/\theta_1 = 3, \quad \alpha = \beta = .05, \quad A = (1 - \beta)/\alpha = 19, \quad B = \beta/(1 - \alpha) = 1/19,$$

we get from (21) $.95 \leqq L(\theta_0) \leqq .983$ and $.05 \leqq L(\theta_1) \leqq .052$. The upper and lower bounds for $E_\theta(r)$ given by (22) are close together for $\theta = \theta_\theta$ and comparatively far apart for $\theta = \theta_1$. Thus for the case $k = 3$ and $\alpha = \beta = .05$, the difference between the upper and lower bounds is $\leqq .06$ for $\theta = \theta_0$ and is about 2.5 for $\theta = \theta_1$.

The left side of (22) is the approximate formula (10) for $E_\theta(r)$ except that the $L(\theta)$ in (22) refers to the exact value and the $L(\theta)$ in (10) is given by the approximation (7). In view of the preceding paragraph, the values of $E_{\theta_0}(r)$ given in Table 1 are very close to the correct values, while the values of $E_{\theta_1}(r)$ are essentially lower bounds for the correct value. We cannot say more unless we go through more extensive calculations of the sort to be described in Remark 2.

Remark 2. Some exact calculations of $L(\theta)$ and $E_\theta(r)$.

Wald ([8], pp. 45–46) pointed out that in order to have a test of exactly strength (α, β), the A and B in (2) should be replaced by A^* and B^*, where $A^* \leqq A = (1 - \beta)/\alpha$ and $B^* \geqq B = \beta/(1 - \alpha)$. In the present case, with information available continuously in time, $B^* = B = \beta/(1 - \alpha)$ since the acceptance of H_0 involves no excess over the boundary. However, acceptance of H_1 does, in general, entail a positive excess over the boundary, and all we can say initially about A^* is that it should lie between $A\ \theta_1/\theta_0$ and A. Thus using $A = (1 - \beta)/\alpha$ instead of A^* is an approximation.

The approximate test based on using A and B is suitable for all practical purposes, since one consequence of the inequalities (21) is that the strength (α', β') is such that $\alpha' \leqq \alpha$, $\beta \leqq \beta' \leqq \beta/(1 - \alpha)$. Since α and β are generally small ($\leqq .10$ say) a procedure based on A and B provides essentially the same protection against errors of the first and second kind as does the test based on using A^* and B^*. However, the use of A rather than A^* in (2) will entail a small increase in $E_\theta(r)$, particularly for $\theta < s$.

As a practical matter, one would usually be content with a test based on (2) which uses A and B. As a matter of fact, this is what is done all the time by people faced with a practical decision problem. For most sequential problems, the problem of finding the A^* and B^* which will give exactly strength (α, β) has not been solved. One has to rely, in such cases, on the results of Wald which indicate that the errors involved in using A, B, and approximate formulae for $L(\theta)$ and $E_\theta(r)$ are "reasonably" small.

In the problem at hand we know, in view of the continuous availability of information, that $B^* = B = \beta/(1 - \alpha)$. Furthermore, formulae are available for computing A^* and for computing O.C. and ASN curves exactly. The formulae for accomplishing these tasks are available [1], [3]. While the computational labor involved in any special case is exceedingly heavy, the results of such computations do throw some light on how exact O.C. and ASN curves compare with those computed by using approximations.

Formulae (4.17) and (4.23) in [3] (similar formulae are given in [1], p. 102) were used to compute

(i) the exact O.C. and $E_\theta(r)$ curves for the semi-continuous rule (2) with $B = \beta/(1 - \alpha)$ and $A = (1 - \beta)/\alpha$. This was done for the case $k = \theta_0/\theta_1 = 3$ and $\alpha = \beta = .05$, and

(ii) A^* (where $A\ \theta_1/\theta_0 \leqq A^* \leqq A$) such that the decision rule

$$(2') \qquad \beta/(1 - \alpha) < (\theta_0/\theta_1)^r \exp[-(1/\theta_1 - 1/\theta_0)\, V(t)] < A^*$$

has an O.C. curve for which $L(\theta_0) = 1 - \alpha$ and $L(\theta_1) = \beta$ exactly, and then to compute $E_\theta(r)$ for the (B, A^*) rule. This was done for the cases $\alpha = \beta = .05$ and $k = \theta_0/\theta_1 = 3/2$, 2, and 3, and also for $\alpha = \beta = .01$ and $k = 3$.

The result of (i) was

$$L(\theta) = .968, \quad L(s) = .529, \quad L(\theta_1) = .051,$$
$$E_{\theta_0}(r) = 3.03, \quad E_s(r) = 8.10, \quad E_{\theta_1}(r) = 7.00.$$

Computation (ii) gave

$\alpha = \beta$	k	A^*	$E_{\theta_0}(r)$	$E_s(r)$	$E_{\theta_1}(r)$
.05	3	13.25	2.94	7.22	6.21
	2	15.1	8.64	18.0	13.8
	$3/2$	16.6	27.9	52.8	36.8
.01	3	68.9	5.00	17.5	10.5

Bearing in mind that the computations were carried through only in a small number of cases, one can make three observations:

(a) For the case $k = 3$ and $\alpha = \beta = .05$, the use of $B = 1/19$ and $A = 19$ results in getting $\alpha' = .032$ and $\beta' = .051$ as compared with $\alpha = \beta = .05$ when one uses $B^* = B = 1/19$ and $A^* = 13.25$. Also, $E_\theta(r)$ is increased by .09, .88, and .79 at $\theta = \theta_0$, s, θ_1 respectively.

(b) Of more interest is the fact that the exact values of $E_\theta(r)$ for the (B, A^*) rule practically coincide with the *approximate* values of $E_\theta(r)$ computed for the (B, A) rule by formulae (10) and (11) and given in Table 1.

(c) In the range of values of $k = \theta_0/\theta_1$ and of α and β covered by Table 1, a good guess at the value of A^* is the value A^{**} lying midway between A and A/k, the upper and lower limits on A^*. This means that $A^{**} = (k + 1)A/2k$. On the basis of our limited calculations we conjecture that in the range of values covered in Table 1, a semi-continuous decision rule (2) with A replaced by A^{**} will have almost exactly strength (α, β). The values of $E_\theta(r)$ associated with a (B, A^{**}) rule will be given to a close approximation by (10).

Remark 3. An approximate formula for $E_\theta(t)$ in the nonreplacement case. A useful approximation to $E_\theta(t)$ in the nonreplacement case is given by $E_\theta(t) \sim \theta \log (n/[n - E_\theta(r)])$. This approximation is obtained by replacing $E_\theta(X_{k,n})$ in (20) by its approximation $\theta \log (n/[n - k])$. Thus (20) becomes

$$E_\theta(t) \sim \theta\, E_\theta \left[\log \left(\frac{n}{n - k} \right) \right] \sim \theta \log \left(\frac{n}{n - E_\theta(r)} \right).$$

This approximation has been tested numerically by calculations on truncated nonreplacement decision procedures, where the exact values of $E_\theta(t)$ can be computed and compared with the suggested approximation. The agreement is close.

5. Numerical examples. These methods have been applied to eight problems.

Problem 1. Find a sequential replacement procedure for testing $H_0 : \theta = \theta_0 = 7500$ hours against $H_1 : \theta_1 = 2500$ hours with $\alpha = \beta = .05$. The constant number of items under test is $n = 100$.

Solution 1 (approximate). The (B, A) test (2) becomes in this case

$$1/19 < 3^r e^{-V(t)/3750} < 19,$$

where $V(t) = 100t$ hours. For this rule $\alpha' = .032$ and $\beta' = .051$.

Solution 2 (exact). The (B, A^*) test $(2')$ becomes

$$1/19 < 3^r e^{-V(t)/3750} < 13.25.$$

For this rule $\alpha = \beta = .05$ exactly.

Problem 2. Compute $E_\theta(r)$ and $E_\theta(t)$ for $\theta = 0$, $\theta_1(= 2500)$, $s(= 4115)$, $\theta_0(= 7500)$, ∞.

Solution. For the (B, A) and (B, A^*) rules, respectively, the values of $E_\theta(r)$ are given in Remark 2 of Section 4. In the replacement case $E_\theta(t)$ is found most easily for all values of $\theta(\neq \infty)$ by using (12), $E_\theta(t) = (\theta/n)E_\theta(r)$.

$$\begin{array}{rlllll} \theta = & 0 & \theta_1 & s & \theta_0 & \\ E_\theta(t) = \Bigg\{ & 0 & 175 & 333 & 227 & \text{using the } (B, A) \text{ rule;} \\ & 0 & 155 & 297 & 220 & \text{using the } (B, A^*) \text{ rule.} \end{array}$$

For $\theta = \infty$, the expected waiting time to reach a decision is given by t_∞, where $e^{-100t_\infty/3750} = 1/19$. This gives $t_\infty = E_\infty(t) = 110$.

Remark. More generally, in terms of B, n, θ_0, and k, we find

$$t_\infty = -\theta_0 \log B \,/\, n(k - 1).$$

This means that if no items fail by t_∞, we stop experimentation at t_∞ with acceptance of H_0.

Problem 3. Assume that we are testing the hypothesis in problem 1 and that we are using the (B, A) rule. A sample of size 100 is placed on test. Items which fail are replaced by new items drawn from the same lot. The experiment is started at time $t = 0$. The first five failures occur at $x_1 = 20.1$ hours, $x_2 = 100.5$ hours, $x_3 = 121.7$ hours, $x_4 = 167.4$ hours, and $x_5 = 179.2$ hours, all times being measured from $t = 0$.

(a) Verify that no decision has been reached by time x_5.

(b) Verify that if the sixth failure has not yet occurred at 287.5 hours, measured from $t = 0$, we can stop experimentation at that time with acceptance of H_0.

Solution. It can be readily verified that, in this case, (5) becomes $-100 + 37.5r < t < 100 + 37.5r$. This region is drawn in Figures 1 and 2. The life test data are plotted by moving vertically so long as we are waiting for the next failure to occur and moving horizontally by one unit (in r) at each failure time. In Figure 1, the path crosses into the region of acceptance, when $r = 5$, at time $t = 100 + (37.5)5 = 287.5$. Since the sixth failure has not yet occurred, we can stop experimentation at $t = 287.5$ with acceptance of H_0.

Remark. As a matter of fact we happen to know in this case that the sixth failure occurs at $x_6 = 346.7$ hours. Thus, as indicated in Figure 1, we saved $346.7 - 287.5 = 59.2$ hours by observing the life test continuously in time.

Problem 4. The first seven failure times in a sample of 100 (with replacement) are $x_1 = 19.3$, $x_2 = 45.8$, $x_3 = 49.9$, $x_4 = 96.7$, $x_5 = 115.2$, $x_6 = 127.7$, and $x_7 = 131.2$. Verify that if the hypotheses being tested are those in Problem 1, then H_0 is rejected at time $x_7 = 131.2$ hours.

Solution. See Figure 2.

Remark. While the *acceptance* in Problem 3 is made *between* failure times x_5 and x_6, the *rejection* in Problem 4 is made *at* failure time x_7, with an excess over the boundary.

Problem 5. Find a truncated (nonsequential) replacement procedure for testing the hypothesis in Problem 1, using a constant sample size $n = 100$.

Solution. From results in [6], it can be verified that the truncated replacement procedure meeting the requirements is:

If min $[x_{10}, 407.5] = 407.5$, truncate the experiment at 407.5 with acceptance of H_0.

If min $[x_{10}, 407.5] = x_{10}$, truncate the experiment at x_{10} with acceptance of H_1.

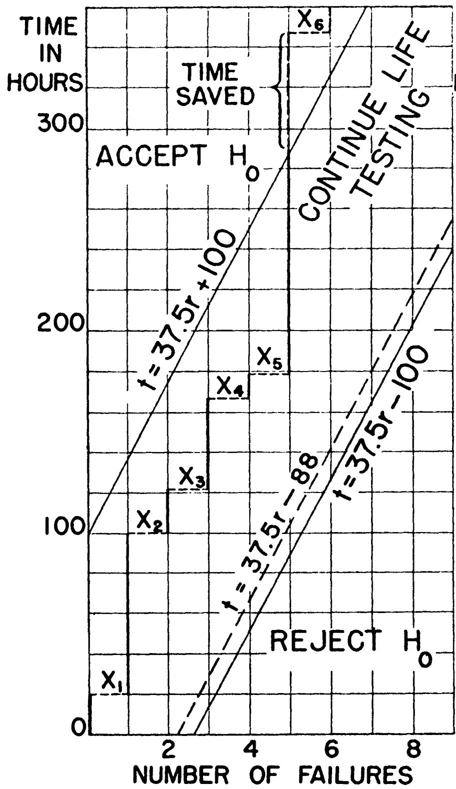

FIG. 1

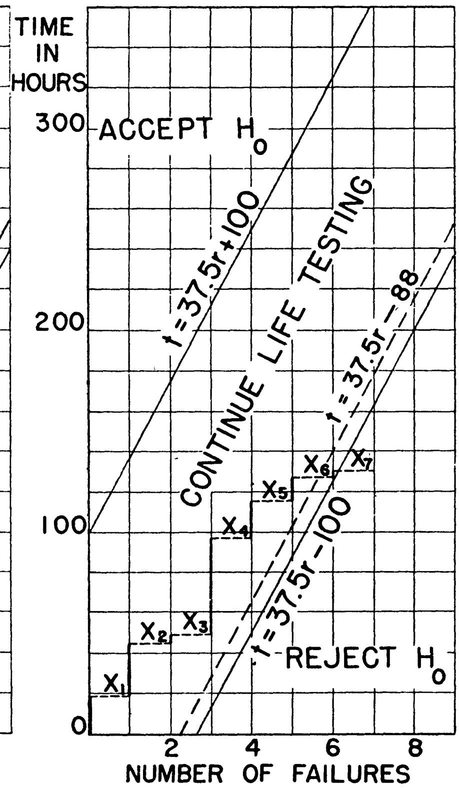

FIG. 2

The solid lines in Figures 1 *and* 2 *give the boundaries of a sequential with replacement test of* $H_0 : \theta_0 = 7500$ *hours against* $H_1 : \theta_1 = 2500$ *hours, with* $\alpha = \beta = .05$, *and* n = *sample size* = 100, *when one uses a* (B, A) *rule. For the* (B, A^*) *rule the upper boundary remains the same, but the lower boundary becomes the dashed line,* $t = 37.5r - 88$. *Figure* 1 *gives a graphical treatment of Problem* 3. *Figure* 2 *gives a graphical treatment of Problems* 4 *and* 8.

The O.C. curves of this test procedure and of the one in Problem 1 are essentially the same.

Problem 6. Compute $E_\theta(r)$ and $E_\theta(t)$ for the plan in Problem 5 for $\theta = 0$, θ_1, s, θ_0, ∞.

Solution. From results in [6], $E_\theta(r) =$ 10, 9.93, 8.75, 5.39, 0, and $E_\theta(t) = (\theta/n)E_\theta(r)$. For $\theta = 0$, θ_1, s, θ_0, ∞, respectively, $E_\theta(t) = 0$, 248, 360, 404.5, 407.5, respectively.

Remark. In Figure 3 we compare the $E_\theta(r)$ and $E_\theta(t)$ curves for Problems 2 (using a B, A^* rule) and 6. This will give some idea of the saving in the expected number of failures and time to reach a decision.

Problem 7. Find t_∞ in Problem 1 if $\alpha = \beta = .01$.

Solution. $t_\infty = -\theta_0 \log B / n(k - 1) = 230$. This is about twice the value of t_∞ when $\alpha = \beta = .05$.

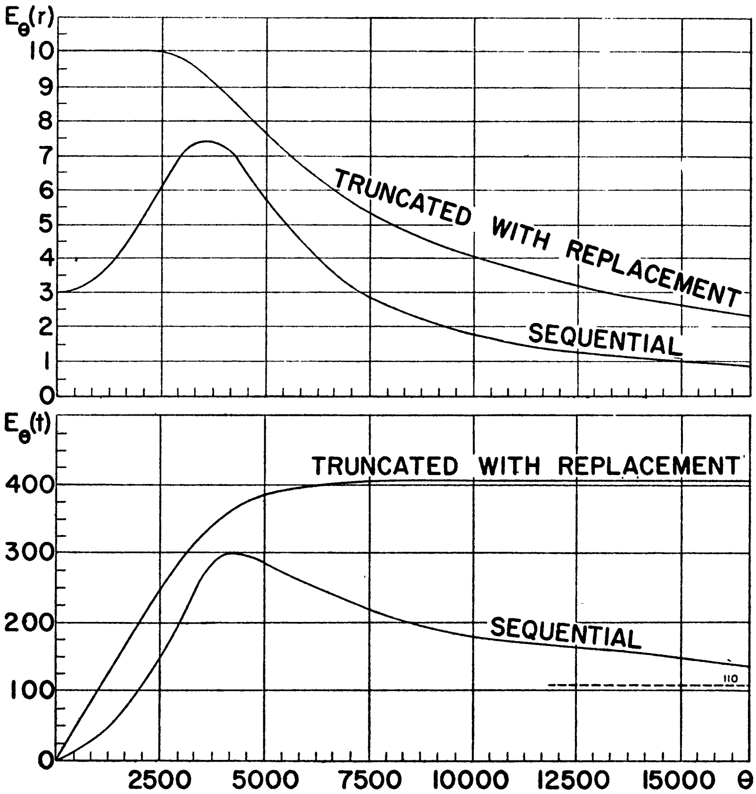

FIG. 3

Comparison of $E_\theta(r)$ (upper portion) and $E_\theta(t)$ (lower portion) curves for sequential and truncated with replacement plans. The O.C. curves for each plan are such that $L(\theta_0) = .95$ and $L(\theta_1) = .05$, with $\theta_0 = 7500$ and $\theta_1 = 2500$. The 110 dashed line gives the value which $E_\theta(t)$ approaches asymptotically as $\theta \to \infty$.

Problem 8. What happens in Problems 3 and 4 if a (B, A^*) rule is used?

Solution. The decision regions are

$$(B, A): \qquad -100 + 37.5r < t < 100 + 37.5r.$$

$$(B = 1/19, A^* = 13.25): \qquad -88 + 37.5r < t < 100 + 37.5r.$$

No change occurs in the solution of Problem 3, since exactly the same decision boundary is being used for accepting H_0. However, the boundary used for rejecting H_0 when using (B, A^*) is shifted by .32 units (in r) to the left in Figures 1 and 2. For the data in Problem 4, this results in rejecting H_0 at time $x_6 = 127.7$ hours, since for $r = 6$, we have $-88 + 37.5r = 137$. This decision to reject H_0 is thus reached with one less item failed and 3.5 hours sooner than in Problem 4.

6. Acknowledgement. We wish to express our appreciation for the work of Mr. Hershel Harrison, who devoted many hours carrying out the calculations, the results of which are summarized in Remark 2 of Section 4.

REFERENCES

[1] J. P. BURMAN, "Sequential sampling formulae for a binomial population", *J. Roy. Stat. Soc., Ser. B*, Vol. 8 (1946), pp. 98–103.

[2] J. L. DOOB, *Stochastic Processes*, John Wiley and Sons, 1953.

[3] A. DVORETSKY, J. KIEFER, and J. WOLFOWITZ, "Sequential decision problems for processes with continuous time parameter. Testing hypotheses", *Ann. Math. Stat.*, Vol. 24 (1953), pp. 254–264.

[4] B. EPSTEIN and M. SOBEL, "Life testing", *J. Amer. Stat. Assoc.*, Vol. 48 (1953), pp., 486–502.

[5] B. EPSTEIN, "Statistical problems in life testing", Proceedings of the Seventh Annual Convention of the American Society for Quality Control, 1953, pp. 385–398.

[6] B. EPSTEIN, "Truncated life tests in the exponential case", *Ann. Math. Stat.*, Vol. 25 (1954), pp. 555–564.

[7] L. H. HERBACH, "Bounds for some functions used in sequentially testing the mean of a Poisson distribution", *Ann. Math. Stat.*, Vol. 19 (1948), pp. 400–405.

[8] A. WALD, *Sequential Analysis*, John Wiley and Sons, 1947.