distribution, given $\bar{x}$, is continuous. We then replace c_δ by $c'_\delta(\bar{x})$, where $c'_\delta(\bar{x})$ is, say, the upper 100δ per cent point of the conditional distribution of Y given $\bar{x}$. From the sufficiency of $\bar{x}$ it follows that $c'_\delta(\bar{x})$ is independent of δ, and the rest of the proof follows through. Under similar circumstances the general theorem proved earlier will remain true in the restricted class U^* of all nonrandomized unbiassed estimators of μ_θ.

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ON CONFIDENCE INTERVALS OF GIVEN LENGTH FOR THE MEAN OF A NORMAL DISTRIBUTION WITH UNKNOWN VARIANCE

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1. Summary. The problem of finding a confidence interval of preassigned length and of more than a given confidence coefficient for the unknown mean of a normal distribution with unknown variance is insoluble if the sample size used is fixed before sampling starts. In this paper two-sample plans, with the size of the second sample depending upon the observations in the first sample (as in [1]), are discussed. Consideration is limited to those schemes which increase the center of the final confidence interval by k if each observation is increased by k, and for which the size of the second sample is a function only of the differences among the observations in the first sample. Then it is shown that the mean of all the observations taken should be used as the center of the final confidence interval. Those schemes which make the size of the second sample a nondecreasing function of the sample variance of the first sample are shown to have certain desirable properties with respect to the distribution of the number of observations required to come to a decision.

2. Assumptions. We deal with an infinite sequence of independent and identically distributed chance variables, $(X_1, X_2, \cdots)$. Each has a normal distribution with unknown mean μ and unknown standard deviation σ. A positive integer $n \geqq 2$ is given, and $X_1, \cdots, X_n$ are observed. Then additional chance

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variables are observed to the number of $N(X_2 - X_1, \; X_3 - X_1, \cdots, X_n - X_1)$. This last expression is a measurable function of $X_2 - X_1, \cdots, X_n - X_1$, which we shall often abbreviate to N.

A fixed positive number Δ and a fixed β, with $0 < \beta < 1$, are given. We are going to study schemes that yield a confidence interval for μ of length Δ, and of confidence coefficient greater than β, regardless of the value of σ. We limit consideration to schemes such that after the second sample has been taken, the center of the final confidence interval is a function $C(X_1, \cdots, X_n, X_{n+1}, \cdots, X_{n+N})$ such that

$$C(X_1 + k, \cdots, X_{n+N} + k) = C(X_1, \cdots, X_{n+N}) + k,$$

identically in k and in those values of $X_1, \cdots, X_{n+N}$ for which we take a total of $n + N$ observations, for all values of N.

3. The optimum center of the final confidence interval. Once a function $N(X_2 - X_1, \cdots, X_n - X_1)$ has been assigned, then the distribution of the total sample size is determined. This distribution does not depend on μ for the class of schemes under consideration. Once we have chosen the function N, there is still the problem of assigning the center of the confidence interval, which will affect the confidence coefficient. (The confidence coefficient is a function of σ and of the function N, but not of μ for the cases we are discussing.) We have

LEMMA 3.1. *For any given function $N(X_2 - X_1, \cdots, X_n - X_1)$, setting the center of the confidence interval at the mean of all the $n + N$ observations taken maximizes the confidence coefficient uniformly in σ.*

PROOF. The joint conditional density of $X_1, \cdots, X_n, X_{n+1}, \cdots, X_{n+m}$, given that $N(X_2 - X_1, \cdots, X_n - X_1) = m$, is equal to

$$H(\sigma) \exp \{-\textstyle\sum_1^{n+m} (x_i - \mu)^2/2\sigma^2\}$$

in the region where $N = m$, and zero elsewhere. The exact form of $H(\sigma)$ is of no concern to us. Denoting by $\bar{X}$ the mean of the values $X_1, \cdots, X_{n+m}$, we can write $C(X_1, \cdots, X_{n+m})$ as $\bar{X} + G(X_1 - \bar{X}, \cdots, X_{n+m} - \bar{X})$, where the function G determines and is determined by the function C. A simple calculation shows that choosing $G(X_1 - \bar{X}, \cdots, X_{n+m} - \bar{X})$ identically equal to zero maximizes

$$P\{(\bar{X} + G - \tfrac{1}{2}\Delta) < \mu < (\bar{X} + G + \tfrac{1}{2}\Delta) \mid (N = m)\}$$

with respect to the function G for any value of σ, thus proving the lemma.

This calculation also shows that once we set $\bar{X}$ as the center of our confidence interval, the conditional probability, given that $N = m$, of having our confidence interval contain μ when σ is the standard deviation is equal to

$$\frac{\sqrt{(n + m)/2\pi}}{\sigma} \int_{-\Delta/2}^{\Delta/2} \exp \{-(n + m)t^2/2\sigma^2\}\, dt, \tag{1}$$

and does not depend on which function $N(X_2 - X_1, \cdots, X_n - X_1)$ is used.

We shall denote the quantity given in (1) by $\beta(\sigma, m)$. We note that $\beta(\sigma, m)$ is increasing in m for any fixed σ, and decreasing in σ for any fixed m. For fixed m, $\beta(\sigma, m)$ approaches unity as σ approaches zero.

4. Properties of a certain class of functions $N(X_2 - X_1, \cdots, X_n - X_1)$. We shall denote $\sum_1^n (X_i - \bar{X})^2$ by $S(X)$, with X standing for the generic point in $(X_1, \cdots, X_n)$ space. The symbol $P(R \mid \sigma)$ stands for the probability of the region R when the standard deviation has the value σ. We have

LEMMA 4.1. *Given any function $N(X_2 - X_1, \cdots, X_n - X_1)$, any non-negative integer m, and any positive value σ', there exists a non-negative number $d(\sigma', m)$ such that*

$$P[S(X) < d(\sigma', m) \mid \sigma] \gtreqless P[N(X_2 - X_1, \cdots, X_n - X_1) \leqq m \mid \sigma]$$

according to whether $\sigma \lesseqgtr \sigma'$.

PROOF. The two sets of values $(X_2 - X_1, \cdots, X_n - X_1)$ and $(X_2 - \bar{X}, \cdots, X_n - \bar{X})$ determine each other uniquely. The joint density of the second set is

$$p(X_2 - \bar{X}, \cdots, X_n - \bar{X}) = (K/\sigma^{n-1}) \exp\{-S(X)/2\sigma^2\}, \qquad K \text{ constant.}$$

From this, it follows easily that if $\sigma_0 < \sigma_1 < \sigma_2$, then of all regions R in $(X_2 - \bar{X}, \cdots, X_n - \bar{X})$ space with $P(R \mid \sigma_1)$ equal to a specified value, the region of the form $S(X) < d$, where d is a constant, minimizes $P(R \mid \sigma_2)$ and maximizes $P(R \mid \sigma_0)$. This proves the lemma.

5. Optimum properties of a certain class of procedures for setting the size of the second sample. For any sample-size rule R, $g(m, \sigma, R)$ denotes the probability that the size of the second sample will be m when the standard deviation is σ and R is used, while $G(m, \sigma, R)$ denotes the probability that the size of the second sample will not exceed m under the same circumstances. Let

$$B(\sigma, R) = \sum_{m=0}^{\infty} g(m, \sigma, R)\, \beta(\sigma, m), \qquad b(\sigma, m, R) = \sum_{i=0}^{m} g(i, \sigma, R)\, \beta(\sigma, i).$$

Thus $B(\sigma, R)$ is the confidence coefficient when R is used and σ is the standard deviation.

We shall say that a rule T setting the size of the second sample is of *type* S if it assigns size m to precisely those points $X = (X_1, \cdots, X_n)$ for which

$$a_{m-1}(T) < S(X) \leqq a_m(T), \qquad m = 0, 1, 2, \cdots,$$

where $a_{-1}(T), a_0(T), \cdots$ are preassigned numbers such that $0 = a_{-1}(T) \leqq a_0(T) \leqq a_1(T) \leqq \cdots$. It is clear that for any rule T of type S, if for some positive σ we have $G(m, \sigma, T)$ approaching unity as m increases, then this happens for all σ.

LEMMA 5.1. *If T is a sample-size rule of type S such that for any finite σ there is a finite positive integer $s(\sigma, T)$ such that $b[\sigma, s(\sigma, T), T] > \beta$, then for any σ' there is a finite positive integer $N(\sigma', T)$ such that $b[\sigma, N(\sigma', T), T] > \beta$ for all $\sigma \leqq \sigma'$.*

PROOF. From continuity considerations, it is easily seen that for any σ in the interval $(0, \sigma']$, if $b[\sigma, s(\sigma, T), T] > \beta$ then σ is in the interior of a nondegenerate interval such that for any σ'' in the interval, $b[\sigma'', s(\sigma, T), T] > \beta$. By the Heine-Borel theorem, a finite number of such intervals can be found to cover the interval $[\delta, \sigma']$ for any positive δ. This proves the lemma, since a neighborhood of zero causes no trouble.

THEOREM 5.1. *Given any sample-size rule R such that $G(m, \sigma, R)$ approaches unity as m increases for any value of σ, and $B(\sigma, R) > \beta$ for all σ, there is a rule R' of type S, with $B(\sigma, R') \geqq \beta$ for all σ, such that for any σ there is a nonnegative finite integer $M(\sigma)$ so that*

$$G(m, \sigma, R') \geqq G(m, \sigma, R), \qquad \textit{for all } m > M(\sigma).$$

PROOF. For the sake of definiteness, if T is any rule of type S, we shall understand that $N(\sigma', T)$ is the smallest positive integer such that $b[\sigma, N[\sigma', T], T] > \beta$ for all $\sigma \leqq \sigma'$.

Let σ_0 be the value of σ which is the solution to $\beta(\sigma, 0) = \beta$. Let δ denote a fixed positive number, and σ_i denote $\sigma_0 + i\delta$, for $i = 1, 2, \cdots$. Let R_0 be the uniquely determined rule of type S such that $g(m, \sigma_0, R_0) = g(m, \sigma_0, R)$ for all m. From the definition of σ_0, for any $\sigma \leqq \sigma_0$ there is a finite integer $s(\sigma, R_0)$ defined in Lemma 5.1.

Also, by Lemma 4.1, $G(m, \sigma, R_0) \leqq G(m, \sigma, R)$ for all $\sigma > \sigma_0$ and for all m. This fact and the assumptions about R imply that for each $\sigma > \sigma_0$ there is a finite $s(\sigma, R)$. We define N_1 as the larger of 1 or $N(\sigma_1, R_0)$. By Lemma 5.1, N_1 is finite.

Next we define another rule R_1 of type S, as follows. Take $a_i(R_1) = a_i(R_0)$ for $i = -1, \cdots, N_1 + 1$, while $a_i(R_1)$ for $i > N_1 + 1$ is chosen so that $G(i, \sigma_1, R_1) = G(i, \sigma_1, R)$. This is possible, since $G(N_1 + 2, \sigma_1, R) \geqq G(N_1 + 2, \sigma_1, R_0)$. It is clear that for any $\sigma \leqq \sigma_1$ there is a finite $s(\sigma, R_1)$. Also, by Lemma 4.1, $G(m, \sigma, R_1) \leqq G(m, \sigma, R)$ for any $\sigma > \sigma_1$. This implies that for any $\sigma > \sigma_1$, there is a finite $s(\sigma, R_1)$. Now we can define the finite number N_2 as the larger of $N_1 + 1$ or $N(\sigma_2, R_1)$.

Next we define a rule R_2 of type S, as follows. Take $a_i(R_2) = a_i(R_1)$ for $i = -1, 0, \cdots, N_2 + 1$, while $a_i(R_2)$ for $i > N_2 + 1$ is chosen so that $G(i, \sigma_2, R_2) = G(i, \sigma_2, R)$. By the same reasoning as above, we see that this is possible, and that we can define a finite $N(\sigma_3, R_2)$. Then we define N_3 as the larger of $N_2 + 1$ or $N(\sigma_3, R_2)$.

In general, once having defined the rule R_j of type S, we define a rule R_{j+1} of type S, as follows. Take $a_i(R_{j+1}) = a_i(R_j)$ for $i = -1, \cdots, N_{j+1} + 1$, while $a_i(R_{j+1})$ for $i > N_{j+1} + 1$ is chosen so that $G(i, \sigma_{j+1}, R_{j+1}) = G(i, \sigma_{j+1}, R)$. Then we define N_{j+2} as the larger of $N_{j+1} + 1$ or $N(\sigma_{j+2}, R_{j+1})$, and proceed as above.

Clearly, for any fixed i, $\lim_{j\to\infty} a_i(R_j)$ exists. As a matter of fact, above a certain j, $a_i(R_j)$ remains constant. Finally, we define a rule R' of type S by $a_i(R') = \lim_{j\to\infty} a_i(R_j)$. For each j, $B(\sigma, R_j) > \beta$ for any σ. Therefore, it is clear from continuity considerations that for any σ, $B(\sigma, R') \geqq \beta$. Also, for any given σ', there is a k so that $\sigma' < \sigma_k$.

Let m be any integer greater than $N_k + 1$. Then there is a nonnegative integer j so that $N_{k+j} + 1 < m \leqq N_{k+j+1} + 1$. Then for all σ, $G(m, \sigma, R_{k+j+1}) = G(m, \sigma, R')$. Also, $G(m, \sigma_{k+j}, R_{k+j}) = G(m, \sigma_{k+j}, R)$. Therefore by Lemma 4.1, $G(m, \sigma', R_{k+j}) \geqq G(m, \sigma', R)$. Now by construction, $G(m, \sigma, R_{k+j+1}) = G(m, \sigma, R_{k+j})$, for all σ. From these relations, we deduce that $G(m, \sigma', R') \geqq G(m, \sigma', R)$. This proves the theorem, for $M(\sigma')$ can be taken as $N_k + 1$.

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