ON THE FOURIER SERIES EXPANSION OF RANDOM FUNCTIONS[1]

By W. L. Root and T. S. Pitcher

Massachusetts Institute of Technology

Problems involving stationary stochastic processes are often treated by approximating the original processes by Fourier series with orthogonal random coefficients.[2] In this paper we justify this technique in certain instances.

We let $x(t)$ denote a real- or complex-valued stochastic process defined for all values of t. We assume the first and second moments of $x(t)$ exist. We write $R(s, t) = E\{x(s)\overline{x(t)}\}$ and in case $R(s, t)$ is a function of $(t - s)$ only (that is, $x(t)$ is stationary in the wide sense), we write $\rho(t - s) = R(s, t)$. We assume everywhere that $E\{x(t)\} = 0$.

We define the stochastic process $x(t)$ to be *periodic* if the random variables $x(t_1)$ and $x(t_1 + T)$ are equal with probability one for all t_1 and some constant T. If $x(t)$ is periodic, then $R(s, t)$ is periodic in each variable. If $x(t)$ is wide-sense stationary, then it is periodic if and only if $\rho(\tau)$ is periodic.

Our first result follows from the theorem due independently to Karhunen and Loève which states: Let $x(t)$ be continuous in the finite interval (a, b), then

$$x(t) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{1}^{n} x_i \psi_i(t)$$

where the $\psi_i(t)$ form an orthonormal system over (a, b) and where $E\{x_i\overline{x_j}\} = \lambda_i\delta_{ij}$ if and only if the $\psi_i(t)$ and the λ_i are a system of eigenfunctions and eigenvalues of the integral equation

$$\int_a^b R(s, t)\overline{\psi(t)}\, dt = \lambda\overline{\psi(s)}.$$

Theorem 1. *Let $x(t)$ be a wide-sense stationary stochastic process continuous in mean square. Then*

$$x(t) = \text{l.i.m.} \sum_{k=-n}^{n} \frac{x_i e^{ik\omega\tau}}{\sqrt{T}}, \qquad \omega = \frac{2\pi}{T},$$

on the interval $(0, T)$, *where the x_i are pairwise orthogonal if and only if $x(t)$ is periodic with period T.*

Proof. From the Karhunen-Loève theorem and the remark above about periodicity it follows that we need to show that

Received August 17, 1954.

[1] The research in this document was supported jointly by the Army, Navy, and Air Force under contract with the Massachusetts Institute of Technology.

[2] Of course, the use of Fourier series can usually be avoided by the use of the spectral representation theorem for stationary processes. See [1] p. 527.

$$\int_0^T R(s, t)\overline{\psi(t)}\,dt = \lambda\overline{\psi(s)}, \qquad 0 \leqq s \leqq T,$$

is satisfied by exponentials when and only when ρ is periodic with period T.

Suppose the ψ_j's are exponentials, that is

$$\psi_j(t) = e^{in_j\omega t} / \sqrt{T}, \qquad \omega = 2\pi/T,\ n_j \text{ an integer.}$$

Then by Mercer's theorem $\sum_j e^{in_j\omega t}e^{-in_j\omega s} / \lambda_j T$ converges uniformly on $[0, T] \times [0, T]$ to $R(s, t)$. Now since the process is stationary, $R(s, t) = \rho(s - t)$. Hence $\sum_j e^{in_j\omega\tau} / \lambda_j T$ converges uniformly to $\rho(\tau)$ in the interval $-T \leqq \tau \leqq T$. Thus $\rho(0) = \rho(T)$ and since $x(t)$ is wide-sense stationary, $E\{|x(t + T) - x(t)|^2\} = \rho(0) - 2R\,[\rho(T)] + \rho(0) = 0$. The converse is obvious.

Davis [2] shows that if $x(t)$ is a wide-sense stationary, continuous-in-mean-square process which has for every $T > 0$ a Fourier series, with random orthogonal coefficients, which converges to $x(t)$ in mean square over $(0, T)$, then $x(t)$ must be the trivial process with $\rho(\tau) =$ const. This assertion follows from Theorem 1. For to satisfy Davis' hypothesis, ρ must satisfy $\rho(t) = \rho(t + T)$, for every $T > 0$, and the only functions with this property are constants.

A somewhat different problem from the expansion of a random function is the statistical representation of a random function. In particular, here we ask for conditions under which a sum of exponentials with orthogonal random coefficients has the same multivariate probability distributions as a given random function. If $x(t)$ is Gaussian, such a statistical representation essentially always exists.

THEOREM 2. *Let $x(t)$ be a stationary Gaussian process with correlation function*

$$\rho(\tau) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{-n}^{n} c_k e^{i\omega k t}, \quad -\frac{T}{2} \leqq t \leqq \frac{T}{2}, \quad \omega = \frac{2\pi}{T}.$$

Then there exists a process $y(t)$ defined by

$$y(t) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{-n}^{n} x_k e^{i\omega k t}$$

where the x_k are complex Gaussian variables satisfying

$$E\{x_k\} = 0, \quad E\{x_k\overline{x_k}\} = c_k, \quad E\{x_k\overline{x_j}\} = 0, \qquad k \neq j,\ x_k = \overline{x_{-k}},$$

which has the same multivariate distributions as $x(t)$ over the interval $0 \leqq t \leqq \frac{1}{2}T$.

PROOF. It is easily verified that if $y_n(t) = \sum_{-n}^{n} x_k e^{i\omega k t}$, then

$$\lim_{\substack{m\to\infty \\ n\to\infty}} E\,\{|y_m(t) - y_n(t)|^2\} = 0,$$

so the process $y(t)$ is defined. Then

$$E\{y(t)\overline{y(t+\tau)}\} = \sum_{k,n=-\infty}^{\infty} E\{x_k \overline{x_n}\}e^{i\omega kt}e^{-i\omega n(t+\tau)} = \sum_{-\infty}^{\infty} \{|x_k|^2\}e^{-ik\omega\tau} = \rho(-\tau),$$

which proves the assertion.

For any $T < \infty$, $\int_{-T}^{T} E(|x(t)|^2)\,dt = 2TE(|x(o)|^2) < \infty$, so that, by Fubini's theorem, $x(t)$ is almost always measurable and square integrable on $(-T, T)$. Hence for almost every sample function $x(t) = \text{l.i.m.} \sum c_n e^{in\pi t/T}$ on the interval $(-T, T)$, where $c_n = (2T)^{-1} \int_{-T}^{T} x(s)e^{\pi nis/T}\,ds$. The process defined by the two preceding equations has the same (a.e.) sample functions as the original on $(-T, T)$. We have

$$\text{(1)} \quad \begin{aligned} E(c_n \overline{c_m}) &= (2T)^{-2} \int_{-T}^{T} ds \int_{-T}^{T} dt\, e^{\pi i(ns-mt)/T}\rho(s-t) \\ &= (4T)^{-1} \int_{-1}^{1} dv\, e^{\pi i(n-m)v} \int_{(v-1)T}^{(v+1)T} du\, e^{im\pi u/T}\rho(u) \end{aligned}$$

where we have made the substitutions $u = s - t$ and then $v = s/T$. If $n = m$, then the integrand is dominated by $2T\rho(o)$, so that $E(|c_n|^2) \leqq \rho(o)$. That is, c_n is square integrable and $E(c_n\overline{c_m})$ exists.

THEOREM 3. *If ρ is integrable on $(-\infty, \infty)$ and $\int_{-\infty}^{\infty} \rho(t)\,dt \neq 0$, then $E(|c_n|^2) = O(1/T) > 0$. If also $n \neq m$, then*

$$\lim_{T\to\infty} \frac{E(c_n\overline{c_m})}{[E(|c_n|^2)E(|c_m|^2)]^{1/2}} = 0.$$

PROOF. The functions f_T defined by

$$f_T(v) = \int_{(v-1)T}^{(v+1)T} du\, e^{m\pi u/T}\rho(u)$$

are uniformly bounded by $\int_{-\infty}^{\infty} |\rho(u)|\,du$. For every v, with $|v| < 1$

$$f_T(v) \to \int_{-\infty}^{\infty} \rho(u)\,du = a \quad \text{as} \quad T \to \infty.$$

Hence, by Lebesgue's theorem applied to (1)

$$\lim\, (4T)E(c_n\overline{c_m}) = \begin{cases} 0 & n \neq m, \\ 2a & n = m. \end{cases}$$

This implies the result.

THEOREM 4. *If ρ is square integrable and r is its Fourier transform, then*

$$\text{(2)} \quad E(c_n\overline{c_m}) = \frac{(-1)^{n+m}}{4\pi T} \int_{-\infty}^{\infty} r\left(\frac{u+n\pi}{2\pi T}\right) \frac{\sin^2 u}{u[u+(n-m)\pi]}\,du.$$

PROOF.

$$\begin{aligned} E(c_n\overline{c_m}) &= \frac{1}{4T}\int_{-1}^{1} dv\, e^{i\pi(n-m)v} \int_{(v-1)T}^{(v+1)T} du\, e^{i\pi mu/T}\left\{\underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A} dw\, e^{-i2\pi uw} r(w)\right\} \\ &= \frac{1}{4T}\int_{-1}^{1} dv\, e^{i\pi(n-m)v} \lim_{A\to\infty}\left\{\int_{(v-1)T}^{(v+1)T} du\, e^{i\pi mu/T} \int_{-A}^{A} dw\, e^{-i2\pi uw} r(w)\right\} \\ &= \frac{(-1)^{m+1}}{4T}\int_{-1}^{1} dv\, e^{i\pi(n-m)v} \lim_{A\to\infty}\int_{-A}^{A} 2r(w)e^{i(m\pi/T-2\pi w)vT}\,\frac{\sin 2\pi wT\, dw}{(m\pi/T - 2\pi w)} \\ &= \lim_{A\to\infty}\frac{(-1)^{m+1}}{2T}\int_{-A}^{A} r(w)\,\frac{\sin 2\pi wT}{(m\pi/T-2\pi w)}\, dw \int_{-1}^{1} e^{i\pi v(n-2wT)}\, dv. \end{aligned}$$

This yields the formula of the theorem. The first step is justified by an application of the Schwarz inequality, the second by the Fubini theorem, and the third by the Lebesgue bounded convergence theorem and the Fubini theorem.

THEOREM 5. *If ρ is square integrable on $(-\infty, \infty)$ and its Fourier transform vanishes almost everywhere in $(-\epsilon, \epsilon)$ for some $\epsilon > 0$, then*

$$\lim_{T\to\infty}\frac{E(c_n\overline{c_m})}{[E(|c_n|^2)E(|c_m|^2)]^{1/2}} = (-1)^{n-m}$$

for all n and m.

PROOF. By the hypothesis, the integrand in the previous theorem vanishes if $-2\pi\epsilon T - n\pi \leqq u \leqq 2\pi\epsilon T - n\pi$ and, for any $\theta < 1$, T can be chosen so large that outside this interval

$$\frac{\sin^2 u}{u[u + (n-m)\pi]} \geqq \frac{\theta \sin^2 u}{u^2}.$$

Since $r(x) \geqq 0$ for almost all x, for large T

$$r\left(\frac{u+n\pi}{2\pi T}\right)\frac{\sin^2 u}{u[u+(m-n)\pi]} \geqq \theta r\left(\frac{u+n\pi}{2\pi T}\right)\frac{\sin^2 u}{u^2}.$$

Integrating this gives $(-1)^{n+m} E(c_n\overline{c_m}) \geqq \theta\, E(|c_n|^2)$. Hence for large enough T,

$$1 \geqq \frac{(-1)^{n+m}E(c_n\overline{c_m})}{[E(|c_n|^2)E(|c_m|^2)]^{1/2}} \geqq \theta,$$

which implies the result.

Instead of holding n and m constant, we can fix the frequencies associated with them, that is, set $n = aT$ and $m = bT$.

THEOREM 6. *If ρ is integrable and square integrable, if a is a real number and $p \neq q$ are integers such that $r(a/2) > 0$ and $r((q/p)(a/2)) > 0$ then*

$$\lim_{T_k\to\infty}\frac{E(c_{pk}\overline{c_{qk}})}{[E(|c_{pk}|^2)E(|c_{qk}|^2)]^{1/2}} = 0, \qquad T_k = \frac{kp}{a}, \qquad k = 0, 1, 2, \cdots.$$

In fact the expression above is $O([(p-q)k]^{-\theta})$ for any $\theta < 1$.

PROOF. By Lebesgue's theorem and the boundedness of r, which follows from the integrability of ρ,

$$\frac{4\pi pk}{a} E(|c_{pk}|^2) = \int_{-\infty}^{\infty} r\left(\frac{au}{pk} + \frac{a}{2}\right) \frac{\sin^2 u}{u^2}\, du \to r\left(\frac{a}{2}\right) \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2}\, du = \pi r\left(\frac{a}{2}\right).$$

Similarly, $(4\pi p^2 k/qa)E(|c_{qk}|^2) \to r(\,(q/p)(a/2)\,\pi)$. Thus

$$\left| \frac{4\pi pk}{a} E(c_{pk}\, \overline{c_{qk}}) \right| = \left| \int_{-\infty}^{\infty} r\left(\frac{au}{2\pi pk} + \frac{a}{2}\right) \frac{\sin^2 u}{u[u + (q-p)k\pi]}\, du \right|$$

$$\leqq \max |r(x)| \int_{-\infty}^{\infty} \left| \frac{\sin^2 u}{u[u + (q-p)k\pi]} \right| du.$$

But the above integral is $O(([q - p]k)^{-\theta})$ for any $\theta < 1$.

THEOREM 7. *If ρ is integrable and square integrable, a is a point at which $r(a/2) > 0$, and p and q are integers, then*

$$\lim_{T_k \to \infty} \frac{E(c_{pk}\, \overline{c_{pk+q}})}{[E(|c_{pk}|^2)E(|c_{pk+q}|^2)]^{1/2}} = 0, \qquad T_k = \frac{kp}{a}, \qquad k = 0, 1, 2, \cdots.$$

PROOF. As above, we get $\lim\, (4\pi pk/a)(E|c_{pk}|^2) = \lim\, (4\pi pk/a)E(|c_{pk+q}|^2) = r(a/2)\pi$. Now

$$\left| \frac{4\pi pk}{a} E(c_{pk}\, \overline{c_{pk+q}}) \right| = \left| \int_{-\infty}^{\infty} r\left(\frac{au}{2\pi pk} + \frac{a}{2}\right) \frac{\sin^2 u}{u(u + q\pi)}\, du \right|$$

$$= \left| \int_{-\infty}^{\infty} \left[r\left(\frac{au}{2\pi pk} + \frac{a}{2}\right) - r\left(\frac{a}{2}\right) \right] \frac{\sin^2 u}{u(u + q\pi)}\, du \right|,$$

since $\int_{-\infty}^{\infty} (\sin^2 u)\, [u(u + q\pi)]^{-1}\, du = 0$ for $q \neq 0$. But by Schwarz's inequality

$$\left| \frac{4\pi pk}{a} E(c_{pk}\overline{c_{pk+q}}) \right|^2 \leqq \int_{-\infty}^{\infty} \frac{\sin^2 v}{v(v + q\pi)^2}\, dv \int_{-\infty}^{\infty} \left| r\left(\frac{au}{2\pi pk} + \frac{a}{2}\right) - r\left(\frac{a}{2}\right) \right|^2 \frac{\sin^2 u}{u^2}\, du.$$

The second integral approaches zero by Lebesgue's theorem.

It is easy to find examples of processes, vanishing around $a/2$ and $b/2$, for which the conclusion of Theorem 6 is false. Suppose, for example, r vanishes in the intervals

$$\left[\left(\frac{a}{2}\right)\left(\frac{p - \epsilon q}{p - \epsilon p}\right),\ \left(\frac{a}{2}\right)\left(\frac{q}{p}\right)\right],$$

$$\left[\left(\frac{qa}{2p}\right)\left(\frac{p}{q}\right)^2,\ \left(\frac{qa}{2p}\right)\left(\frac{q - p\epsilon}{p - p\epsilon}\right)\right], \qquad q > p > 0, \epsilon > 0.$$

Then $\epsilon E(|c_{pk}|^2) - |E(c_{pk}\overline{c_{qk}})| < 0$ and $\epsilon E(|c_{qk}|^2) - |E(c_{qk}\overline{c_{pk}})| < 0$.

Since the coefficients corresponding to frequencies x with $r(x) = 0$ tend to misbehave, it is desirable to show that the total effect of such coefficients is small for large T. The following theorem does this for the band limited case.

THEOREM 8. *If r is bounded and vanishes outside $(-A/2, A/2)$ for some A, then*

$$\left[\sum_{[AT]+1}^{\infty} E(|c_n|^2) + \sum_{-[AT]+1}^{-\infty} E(|c_n|^2)\right] \Big/ \sum_{-\infty}^{\infty} E(|c_n|^2) = 0\left(\frac{\log T}{T^2}\right),$$

where $[AT]$ *is the largest integer not exceeding* AT.

PROOF.

$$\sum_{-\infty}^{\infty} E(|c_n|^2) = E\left(\int_{-T}^{T} |x(t)|^2\, dt\right) = \int_{-T}^{T} E(|x(t)|^2)\, dt = 2T\rho(o),$$

$$\sum_{[AT]+1}^{\infty} E(|c_n|^2) = \sum_{[AT]+1}^{\infty} \frac{1}{4\pi T} \int_{-\infty}^{\infty} r\left(\frac{u + n\pi}{2\pi T}\right) \frac{\sin^2 u}{u^2}\, du$$

$$= \frac{1}{2} \sum_{[AT]+1}^{\infty} \int_{-\infty}^{\infty} r(v)\, \frac{\sin^2 2\pi vT}{(2\pi vT + n\pi)^2}\, dv.$$

Now,

$$\sum_{[AT]+1}^{\infty} \frac{1}{(2\pi vT + n\pi)^2} = \frac{1}{\pi^2} \sum_{0}^{\infty} \frac{1}{(2vT + [AT] + 1 + n)^2}$$

$$= \frac{1}{\pi^2} (\log \Gamma)''(2vT + [AT] + 1),$$

$$\sum_{[AT]+1}^{\infty} E(|c_n|^2)\, \frac{1}{2\pi^2} \int_{-A/2}^{A/2} r(v) \sin^2 2\pi Tv (\log \Gamma)''(2vT + [AT] + 1)\, dv$$

$$\leqq \frac{1}{2\pi^2} \sup (r) \int_{-A/2}^{A/2} (\log \Gamma)''(2vT + [AT] + 1)\, dv$$

$$= \frac{\sup (r)}{4\pi^2 T} \int_{[AT]+1-AT}^{[AT]+1+AT} (\log \Gamma)''(\omega)\, d\omega$$

$$\leqq \frac{\sup (r)}{4\pi^2 T} (\log \Gamma)'(\omega) \Big|_{[AT]-AT+1}^{[AT]+AT+1} = O\left(\frac{\log T}{T}\right).$$

The other case is handled similarly.

REFERENCES

[1] J. L. DOOB, *Stochastic Processes*, John Wiley and Sons, New York, 1953.

[2] R. C. DAVIS, "On the Fourier expansion of stationary random processes," *Proc. Amer. Math. Soc.*, Vol. 4 (1953), pp. 564–569.