NOTES

A NECESSARY AND SUFFICIENT CONDITION FOR ADMISSIBILITY

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1. Summary. In Section 2 we give the usual definition for admissibility of a strategy in a two-person zero-sum game, and obtain a simple sufficient condition for admissibility of a strategy for the second player which is hardly more than a formal statement of a procedure frequently used in proving admissibility. In Section 3 we introduce the notion of strict admissibility, which is slightly stronger than admissibility, but equivalent to it in the case where the space of strategies of the second player is weakly compact in the sense of Wald. We then obtain a necessary and sufficient condition for strict admissibility, in the form of a condition on the upper values of a sequence of games associated with the original game. In Section 4 we show that, under the additional condition that the minimax theorem holds for certain associated games, the condition of Section 2 is necessary as well as sufficient. The results have a formal resemblance to those of Hodges and Lehmann [4].

2. Introduction. Let A and B be sets and K a real-valued function on $A \times B$ such that for every $a \varepsilon A$

(1) $$\rho(a) = \inf_{b \varepsilon B} K(a,b) > -\infty.$$

Following von Neumann [1], we refer to the triple (A, B, K) as a two-person zero-sum game, having in mind the situation where the first player chooses an element a of A, the second player an element b of B, the two choices being made simultaneously, and then the second player pays the first player the amount $K(a, b)$. The set B is partially ordered by the relation $\lesssim$ where $b_1 \lesssim b_2$ means that, for every $a \varepsilon A$,

(2) $$K(a, b_1) \leqq K(a, b_2).$$

If this holds we say that b_1 *is better than* b_2. If, for all a,

(3) $$K(a, b_1) = K(a, b_2),$$

we write $b_1 \approx b_2$ and say that b_1 *is equivalent to* b_2. If $b_1 \lesssim b_2$ but not $b_1 \approx b_2$, we write $b_1 \prec b_2$ and say that b_1 *is strictly better than* b_2. We say that b_1 is *admissible* if there exists no b_2 strictly better than b_1.

We shall need a few more definitions before we can indicate the principal result of this paper. The strategy b_1 is said to be ϵ-*Bayes* with respect to $a \varepsilon A$ if

(4) $$K(a, b_1) \leqq \inf_b K(a, b) + \epsilon,$$

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and to be *Bayes* if this holds for $\epsilon = 0$. If $\mathfrak{A}$ is a σ-algebra of subsets of A such that for each b, $K(\cdot, b)$ is $\mathfrak{A}$ measurable, and Ξ is a convex set of probability measures on $\mathfrak{A}$, including at least all those measures, denoted by $[a]$, concentrated at a single point $a \varepsilon A$, then the game (Ξ, B, K') with

$$K'(\xi, b) = \int K(a, b)\, d\xi(a) \tag{5}$$

will be called a *convex extension* of K. In order to make sure that this integral is defined we must make an additional assumption on K, and we shall assume K bounded below. A reasonable alternative might be the condition symmetric to (1), that is,

$$\sup_a K(a, b) < \infty \qquad \text{for all } b. \tag{6}$$

THEOREM 1. *If b_1 is such that for every $a_1 \varepsilon A$ and $\epsilon > 0$ there exists $\xi \varepsilon \Xi$ and $\delta > 0$ such that b_1 is $\epsilon\delta$-Bayes with respect to $(1 - \delta)\xi + \delta[a_1]$, then b_1 is admissible.*

PROOF. Suppose b_1 is not admissible. Then there must exist b_2 which is strictly better than b_1, that is,

$$K(a, b_2) \leqq K(a, b_1) \tag{7}$$

for all a with strict inequality for some a, say a_1. By assumption, there exists $\delta > 0$ and $\xi \varepsilon \Xi$ such that

$$\begin{aligned} K'((1 - \delta)\xi + \delta[a_1], b_1) &\leqq \inf_b K'((1 - \delta)\xi + \delta[a_1], b) + \epsilon\delta \\ &\leqq K'((1 - \delta)\xi + \delta[a_1], b_2) + \epsilon\delta \\ &\leqq (1 - \delta)K(\xi, b_1) + \delta K(a_1, b_2) + \epsilon\delta \end{aligned}$$

so that $K(a_1, b_1) \leqq K(a_1, b_2) + \epsilon$. Since ϵ is arbitrary, $K(a_1, b_1) \leqq K(a_1, b_2)$, which contradicts the hypothesis that (7) holds with strict inequality at a_1.

This theorem essentially follows the reasoning used by Blyth [2] and other authors in proving admissibility. In Section 4, assuming weak compactness of B in the sense of Wald [3], and assuming the minimax theorem to hold for a class of games associated with K', we shall show that this condition is also necessary. The set B is said to be *weakly compact* with respect to K in the sense of Wald if, for every sequence $\{b_i\}$, there exists b_0 and a subsequence $\{b_{i_j}\}$ such that

$$\varliminf_{j \to \infty} K(a, b_{i_j}) \geqq K(a, b_0). \tag{8}$$

We observe that, by Fatou's Lemma, if B is weakly compact with respect to K, and K is bounded below, then B is weakly compact with respect to K'.

The necessity of the condition of Theorem 1 could perhaps be proved more quickly without the intermediate results of Section 3. However, the necessary and sufficient condition, valid under much weaker conditions which we obtain there, is likely to be of some interest.

3. A necessary and sufficient condition for admissibility. In this section we shall use the notation and assumptions of Section 2 through (2.4), and also the definition of weak compactness (2.8). We shall also need the notion of strict admissibility, slightly stronger than admissibility. The strategy b_1 is said to be *strictly admissible* if for every $a_1 \varepsilon A$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for every b for which $K(a_1, b) \leqq K(a_1, b_1) - \epsilon$, there exists a such that $K(a, b) \geqq K(a, b_1) + \delta$. It is clear that strict admissibility implies admissibility.

THEOREM 2. *If b_0 is admissible and B is weakly compact with respect to K, then b_0 is strictly admissible.*

PROOF. Suppose B is weakly compact in the sense of Wald and b_0 is not strictly admissible. Then for some $a_0 \varepsilon A$ and some $\epsilon > 0$ there exists a sequence $\{b_i\}$ such that

$$K(a_0, b_i) \leqq K(a_0, b_0) - \epsilon \qquad \text{for all } i = 1, 2, \cdots, \tag{1}$$

$$\lim_{i\to\infty} \sup_{a \varepsilon A} [K(a, b_i) - K(a, b_0)] \leqq 0. \tag{2}$$

By the assumption of weak compactness there exists a subsequence $\{b_{i_j}\}$ and an element b' such that

$$\underline{\lim}_{j\to\infty} K(a, b_{i_j}) \geqq K(a, b') \quad \text{for all} \quad a. \tag{3}$$

It follows that

$$\begin{aligned} \sup_{a\varepsilon A} [K(a, b') - K(a, b_0)] &\leqq \sup_{a \varepsilon A} [\underline{\lim}_{j\to\infty} K(a, b_{i_j}) - K(a, b_0)] \\ &= \sup_{a\varepsilon A} \underline{\lim}_{j\to\infty} [K(a, b_{i_j}) - K(a, b_0)] \\ &\leqq \underline{\lim}_{j\to\infty} \sup_{a\varepsilon A} [K(a, b_{i_j}) - K(a, b_0)] \leqq 0. \end{aligned} \tag{4}$$

Similarly,

$$K(a_0, b') \leqq K(a_0, b_0) - \epsilon, \tag{5}$$

so that b' is strictly better than b_0. Thus b_0 is not admissible.

THEOREM 3. *In order that b_0 be strictly admissible, it is necessary and sufficient that for every a_0*

$$\overline{\lim}_{\gamma\to\infty} \inf_b \sup_a \{K(a_0, b) - K(a_0, b_0) + \gamma[K(a, b) - K(a, b_0)]\} \geqq 0. \tag{6}$$

In order to simplify the writing we assume without essential loss of generality that, for all a, $K(a, b_0) = 0$. Then (6) becomes

$$\overline{\lim}_{\gamma\to\infty} \inf_b \sup_a \{K(a_0, b) + \gamma K(a, b)\} \geqq 0. \tag{7}$$

Also a_0 may be taken as fixed throughout the proof.

PROOF OF NECESSITY. Suppose b_0 strictly admissible and let δ_ϵ be the δ whose existence is asserted in the definition of strict admissibility. Let S_ϵ be the set of all b such that

$$K(a_0, b) < -\epsilon \tag{8}$$

and S'_ϵ its complement. Then

$$\begin{aligned}
&\underline{\lim}_{\gamma\to\infty} \inf_b \sup_a [K(a_0, b) + \gamma K(a, b)] \\
&\qquad = \underline{\lim}_{\gamma\to\infty} \min \{ \inf_{b \varepsilon S_\epsilon} [K(a_0, b) + \gamma \sup_a K(a, b)], \\
(9)\quad &\inf_{b \varepsilon S'_\epsilon} [K(a_0, b) + \gamma \sup_a K(a, b)]\} \\
&\qquad \geqq \lim_{\gamma\to\infty} \min \{\rho(a_0) + \gamma\delta_\epsilon, \quad -\epsilon + \gamma \inf_b \sup_a K(a, b)\} \geqq -\epsilon.
\end{aligned}$$

The last step follows from the fact that, by the admissibility of b_0, for every b there exists a such that $K(a, b) \geqq 0$, so that

$$\inf_b \sup_a K(a, b) \geqq 0.$$

Since ϵ was arbitrary, this completes the proof of necessity.

PROOF OF SUFFICIENCY. Supposing that (7) holds we have for every $\epsilon > 0$,

$$\begin{aligned}
0 &\leqq \overline{\lim}_{\gamma\to\infty} \inf_b \sup_a [K(a_0, b) + \gamma K(a, b)] \\
&\leqq \overline{\lim}_{\gamma\to\infty} \inf_{b \varepsilon S_\epsilon} \sup_a [K(a_0, b) + \gamma K(a_0, b)] \\
&\leqq \overline{\lim}_{\gamma\to\infty} [\sup_{b \varepsilon S_\epsilon} K(a_0, b) + \gamma \inf_{b \varepsilon S_\epsilon} \sup_a K(a, b)] \\
&= \lim_{\gamma\to\infty} [-\epsilon + \gamma \inf_{b \varepsilon S_\epsilon} \sup_a K(a, b)].
\end{aligned} \tag{10}$$

Consequently, there exists $\gamma_\epsilon > 0$ such that

$$-\tfrac{1}{2}\epsilon \leqq -\epsilon + \gamma_\epsilon \inf_{b \epsilon S_\epsilon} \sup_a K(a, b). \tag{11}$$

Thus the definition of strict admissibility is satisfied with $\delta = \frac{1}{2}\epsilon/\gamma_\epsilon$.

The proof shows that (6) could have been stated with $\overline{\lim}$ replaced by $\underline{\lim}$ or by lim, or with $\geqq$ replaced by $=$, or both.

COROLLARY. *If* (6) *holds for all* a_0, *then* b_0 *is admissible. If* B *is weakly compact, then the converse holds.*

This is an immediate consequence of Theorems 2 and 3.

4. Admissibility in the presence of the minimax theorem. In this section, we suppose K is bounded below and possesses a convex extension (Ξ, B, K') as described around (2.5). We shall also suppose the minimax theorem applies in (2.6) when K is replaced by K', that is,

$$\begin{aligned}
(12)\quad &\inf_b \sup_\xi \{K(a_0, b) - K(a_0, b_0) + \gamma[K'(\xi, b) - K'(\xi, b_0)\} \\
&\qquad = \sup_\xi \inf_b \{K(a_0, b) - K(a_0, b_0) + \gamma[K'(\xi, b) - K'(\xi, b_0)\}.
\end{aligned}$$

THEOREM 4. *Under the above conditions, in order that b_0 be admissible it is necessary and sufficient that for every a_0 and every $\epsilon > 0$ there exist $\xi_1 \varepsilon \Xi$ and $\delta > 0$ such that b_0 is $\epsilon\delta$-Bayes with respect to* $(1 - \delta)\xi_1 + \delta[a_0]$.

PROOF. Using the fact that

$$\sup_{\xi} [K'(\xi, b) - K'(\xi, b_0)] = \sup_{a} [K(a, b) - K(a, b_0)] \tag{2}$$

together with (1), we find that (3.6) is equivalent to

$$\lim_{\gamma \to \infty} \sup_{\xi} \inf_{b} \{K(a_0, b) - K(a_0, b_0) + \gamma[K'(\xi, b) - K'(\xi, b_0)]\} \geqq 0. \tag{3}$$

If we let $\delta = 1/(\gamma + 1)$ and use the fact that

$$\frac{\gamma}{1 + \gamma} K'(\xi, b) + \frac{1}{1 + \gamma} K'([a_0], b) = K'\left(\frac{\gamma}{1 + \gamma} \xi + \frac{1}{1 + \gamma} [a_0], b\right), \tag{4}$$

we find that (3) is equivalent to

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \sup_{\xi} \inf_{b} [K'((1 - \delta)\xi + \delta[a_0], b) - K'((1 - \delta)\xi + \delta[a_0], b_0)] \geqq 0. \tag{5}$$

This is equivalent to the assertion that for every $\epsilon > 0$ there exist $\delta > 0$ and $\xi_1 \varepsilon \Xi$ such that

$$\inf_b K'((1 - \delta)\xi_1 + \delta[a_0], b) \geqq K'((1 - \delta)\xi_1 + \delta[a_0], b_0) - \epsilon\delta. \tag{6}$$

The theorem follows immediately.

REFERENCES

[1] J. VON NEUMANN, "Zur Theorie der Gesellschaftsspiele," *Math. Ann.*, Vol. 100 (1928), pp. 295–320.
[2] C. BLYTH, "On minimax statistical decision procedures and their admissibility," *Ann. Math. Stat.*, Vol. 22 (1951), pp. 22–42.
[3] A. WALD, *Statistical Decision Functions*, John Wiley and Sons, New York, 1950.
[4] J. HODGES, AND E. LEHMANN, "The use of previous experience in reaching statistical decisions," *Ann. Math. Stat.*, Vol. 23 (1952), pp. 396–407.

Note added in proof. I believe this theorem to be potentially useful, but cannot now give any non-trivial examples. Attempts to apply the sufficiency often run into analytic difficulties. The necessity was useful heuristically in the recognition of the inadmissibility of the usual estimate of the mean of a multivariate normal distribution of dimension greater than or equal to 3. (Abstract in *Ann. Math. Stat.*, Vol. 26 (1955), p. 157; to appear in the *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*). A result similar to Theorem 4 has been obtained independently by LeCam.