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ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Berkeley meeting of the Institute, July 14–16, 1955)

1. **Nonparametric Mean Estimation of Percentage Points and Density Function Values.** John E. Walsh, Lockheed Aircraft Corporation.

Consider a sample of size n from a statistical population with probability density function $f(x)$ and $100p$ per cent point θ_p. The function $f(x)$ is of an analytic nature. Some methods are presented for approximate nonparametric expected value estimation of θ_p and of $1/f(\theta_p)$. A nonparametric estimate whose expected value differs from θ_p by terms of order $n^{-7/2}$ can be obtained. For $1/f(\theta_p)$, an estimate whose expected value is accurate to terms of order n^{-3} can be obtained. The estimates developed consist of linear functions of specified order statistics of the sample. The order statistics used are sample percentage points with percentage values which are near $100p$. Let m be the number of order statistics appearing in an estimate ($m \leqq 7$). Coefficients for the linear estimation function are obtained by solving a specified set of m linear equations in m unknowns. All estimates derived for θ_p have variances of the form $p(1 - p)/nf(\theta_p)^2 + 0(n^{-3/2})$. Without additional information, all that can be determined about the variances of the estimates derived for $1/f(\theta_p)$ is that they are $0(n^{-1/2})$. Thus both types of estimates are consistent but the estimates for θ_p are more efficient than those for $1/f(\theta_p)$.

2. **On the Concept of Probability in Quantum Mechanics.** A. O. Barut, Stanford.

Some mathematical consequences of the following particular probability measure are discussed: Consider the one to one correspondence between the elements of the sample space Ω and the linearly independent elements of a unitary space $\vee$ (in general a Hilbert space). The probability measure of sets in Ω is defined by $p(S) = (P_S x, x) = \| P_S x \|^2$, where $(x, x) = 1$ and P_S is the projection operator on the manifold spanned by vectors corresponding to the points in S. The vector x characterizes the system or the experiment. It follows from $p(S)$ that random variables are represented by linear Hermitian operators. These random variables may have an intrinsic correlation coefficient even though they are independent in the ordinary sense; they apply to a larger class of phenomena.

3. **Two-Sample Estimates of Prescribed Precision. (Preliminary Report.)** Allan Birnbaum, Columbia University and Stanford University.

Let $x_1, x_2, \cdots$ be independent observations on a random variable X with density (or discrete probability) function $f(x, \theta)$, with θ unknown, $\theta \in \Omega$, $E(X) = \mu = \mu(\theta)$, Var $(X) = \sigma^2(\theta)$. Suppose an unbiased estimate of μ is required, with variance not exceeding a given positive v. Let $u = u(x_1, \cdots x_m)$ be such that $E(u) \leqq 1/\sigma^2(\theta)$. Let $n = 1/uv$. Let $n' = [n]$ = the largest integer not exceeding n, with probability γ_n, and $n' = [n] + 1$, with probability $1 - \gamma_n$, where γ_n is determined so that $E(1/n' \mid n) = 1/n$. Then the estimate $\mu' = (x_{m+1} + \cdots + x_{m+n'})/n'$ satisfies the requirements. If $E(u) = 1/\sigma^2(\theta)$, then Var $(\mu') = v$, and the estimates μ' satisfy homoscedasticity assumptions of standard statistical methods; such functions u are given for estimating variances of normal distributions and components of variance. For estimating binomial and Poisson means, approximate homoscedasticity is obtained by use of $u = c/(\Sigma_1^m x_i + 1)(m - \Sigma_1^m x_i + 1)$ (binomial) and $u = c/(\Sigma_1^m x_i + 1)$ (Poisson), where c is a suitable constant. Efficiencies, measured by the Wolfowitz-Cramer-Rao bound, are good in many cases. Relative to single sample estimates, appreciable savings are obtained in binomial estimation except for p near $\frac{1}{2}$. Refined estimates based on complete statistics are being investigated. Dr. W. C. Healy has just informed the writer that most of the above ideas were recently found independently by him.

4. **Minimax Test for the Parameter of a Poisson Process.** J. V. Breakwell, North American Aviation, Inc.

The problem of determining all the Bayes procedures for distinguishing between two Poisson process rates, when the losses due to accepting the wrong rate are known and when the cost of observation is proportional to the observation time, is given an explicit solution. The selection of the minimax procedure is explained. The solution is given for the extended problem in which a process rate can have any value and it is desired to decide whether this rate is above or below some critical rate, and in which the loss due to a wrong decision is proportional to the difference between the actual rate and the critical rate. In some situations the minimax procedure is a mixed strategy. For these situations an attempt is also made to obtain the minimax pure strategy.

5. **Distribution of a Definite Quadratic Form in Noncentral Normal Variates.** James Pachares, Hughes Aircraft Company.

The results of a recent paper by the author, "Note on the distribution of a definite quadratic form," *Ann. Math. Stat.*, Vol. 26 (1955), pp. 128–131, have been extended to the noncentral case. That is, an expression has been derived for the distribution of a definite quadratic form in noncentral independent normal variates which depends only on the value of the determinant of the form and on the moments of the inverse quadratic form in normal variates with imaginary means. Theorem. Let $Q_n = \frac{1}{2} \Sigma_{j=1}^n a_j x_j^2$, where the x_j are independent $N(\mu_j, 1)$ variates, $a_j > 0$. Let $Q_n^{**} = \frac{1}{2}\Sigma_{j=1}^n a_j^{-1} y_j^2$, where the y_j are independent $N(i\mu_j, 1)$ variates, $i = \sqrt{-1}$. Then $\Pr(Q_n \leqq t) = e^{-\lambda} t^{n/2} \mid A \mid^{-1/2} \Sigma_{k=0}^{\infty} (-t)^k c_k / k! \Gamma(k + 1 + n/2)$, where c_k is the k-th moment of Q_n^{**}, $\lambda = \frac{1}{2}\Sigma_{j=1}^n \mu_j^2$, and $\mid A \mid = a_1 \cdots a_n$. The above series is absolutely convergent. The moments of Q_n^{**} are best obtained from the cumulants. The k-th cumulant of Q_n^{**} is $\frac{1}{2}(k - 1)! \Sigma_{j=1}^n a_j^{-k}(1 - k\mu_j^2)$.

6. **Errors in Normal Approximations to Certain Types of Distribution Functions. (By Title.)** J. T. Chu, University of North Carolina.

Suppose that for every integer $n \geqq n_0 \geqq 1$, $F_n(x) = C_n \int_{-\infty}^{x} (1 \pm y^2/n)^{\mp m/2} \, dy$ is a cdf, where C_n and m depend only on n and $\lim_{n\to\infty} m/n = 1$. Upper and lower bounds are obtained for $F_n(x)$ in terms of $\Phi(x)$, the standard normal cdf, and it is shown that $\lim_{n\to\infty} F_n(x) = \Phi(x)$ for every fixed x. Applications are given to the t-distribution, τ-distribution, the distributions of the correlation coefficients, etc. Very simple upper bounds are derived for the errors in using the normal approximation. For example, if $F_n(x)$ is the cdf of the t-distribution with n degrees of freedom, then $| F_n(x)/\Phi(x) - 1 | < 1/n$ for all $n \geqq 8$. (Work supported by the Office of Naval Research.)

7. Estimation of the Parameters of the One- and Two-Parameter Single Exponential Distributions from Singly and Doubly Censored Samples (By Title). A. E. Sarhand and B. G. Greenberg, University of North Carolina.

In biological data involving life-testing and incubation periods, the response pattern usually follows an exponential distribution. Owing to the speed of the initial reaction and the drawn out waiting time for the ultimate observation, samples are frequently censored at both termini. Tables of coefficients are provided to calculate the best linear estimate in such censored samples up to size 10 of both μ and σ in the two distributions $f(y) = e^{-y/\sigma}/\sigma$ and $f(y) = e^{-(y-\mu)/\sigma}/\sigma$. Variances of these estimates and their efficiencies relative to the uncensored sample are also provided. Combining this information with a table of expected waiting times, the efficiency of an experiment per unit of waiting time can be plotted against each observation and the censoring procedure carried out on a predetermined basis.

8. The Joint Distribution of Serial Correlation Coefficients. (By Title.) G. S. Watson, Australian National University.

Quenouille (*Ann. Math. Stat.*, Vol. 20 (1949), p. 561) found the exact joint distribution of the serial correlation coefficients (with circular definitions) in samples of an odd number of observations and gave an approximation to it without proof. In this paper, Quenouille's distribution is derived for arbitrary sample sizes by a method which avoids the calculus of residues of several complex variables. It is further shown that Quenouille's conjectured approximation is correct and that it arises from the smoothing of the convex polyhedral region of joint variation of the full set of serial correlations. A test for independence based on periodogram ordinates, is suggested.

9. On Tests of Certain Hypotheses Invariant Under the Full Linear Group. (Preliminary Report.) Charles M. Stein, Stanford University.

There exist two probability measures μ_1, μ_2 absolutely continuous with respect to ordinary Lebesgue measure on the Cartesian product C^3 of three circles (circumferences) C and a measurable set $S \subset C^3$ such that $\beta = \inf_T \mu_2(T'S) > \sup_T \mu_1(T'S) = \alpha$, where T ranges over the group of homeomorphisms of C onto itself and $T'(\theta_1, \theta_2, \theta_3) = (T\theta_1, T\theta_2, T\theta_3)$. Of course the result remains true if T is restricted to be a projective transformation (with a particular homeomorphic identification of C with the real projective line). Thus, for testing (with a single observation) the hypothesis that a random point X of C^3 is distributed so that, for some projective transformation T, $T'X$ has distribution μ_1, against the hypothesis that for some T, $T'X$ has distribution μ_2, the rejection region S has minimum power β greater than its maximum size α. However the induced group of T' on C^3 is transitive on a set whose complement has Lebesgue measure 0. Thus any test of size α invariant under the projective group must also have power α. Since the group of projective transformations is a homomorphic image of the multiplicative group of all non-singular real 2×2 matrices, the result indicates that, if the classical tests in multivariate analysis have any exact minimax properties in the class of all tests, it must be due to special properties of the normal distribution, rather than to the group-theoretical structure alone.

10. Asymptotic Formulae for the Distribution of Hotelling's Generalized T_0^2 Statistic. KOICHI ITO, University of North Carolina.

In the analysis of variance for the means of k p-variate normal populations, let S_0 and S_1 be sample "within" and "between" dispersion matrices based on n and m degrees of freedom, respectively, where it is assumed that $p \leqq n$ but m may be $\geqq p$ or $< p$. Hotelling defines a statistic T_0^2 for testing equality of the k population mean vectors as $T_0^2 = m \text{ tr } S_1S_0^{-1}$. In this paper any percentage point of the T_0^2 distribution under the null hypothesis is expressed in an asymptotic series for large values of n, which involves the corresponding percentage point of the χ^2 distribution with mp degrees of freedom. This result generalizes the asymptotic formula for the generalized Student T given by Hotelling and Frankel (*Ann. Math. Stat.*, Vol. (1938), p. 96). An asymptotic formula for the c. d. f. of T_0^2 is also given together with an upper bound for the error committed when all but the first few terms are omitted in the series. This formula is a sort of multivariate analogue of Hartley's formula of "Studentization" (*Biometrika*, (1944), pp. 173–180).

11. Asymptotic Distribution of Maximum Likelihood Estimates in Factor Analysis in the Loading-Normalized Case. HERMAN RUBIN, Stanford University.

If one considers the factor analysis problem with $x_{ij} = \Sigma\lambda_{i\alpha} f_{\alpha j} + u_{ij}$, where the f's have covariance matrix M and the u's have diagonal covariance matrix Σ and are independent of the f's, the maximum likelihood estimates of Λ, M, and Σ were derived by Anderson and Rubin in a paper to appear in *Proceedings of the Third Berkeley Symposium on Probability and Statistics*. By suitable matrix manipulation, we obtain a symmetric modified Newton's method for $\Sigma^{-1}(\hat{\Lambda} - \Lambda_0)$ and $(\hat{\Sigma}^{-1} - \Sigma_0^{-1})$, starting from the initial approximation Λ_0, Σ_0. If the normalization is on Λ alone, then the normality of the f's can be relaxed and the asymptotic covariance of the estimates is given by the inverse of the coefficient matrix. If we relax the condition of normality of the u's and consider, instead of $\hat{\Sigma} - \Sigma$, the matrix $\hat{\Sigma} - S$, where S is the diagonal matrix of sample variances of the u's, the asymptotic variances of the σ_{ii} are reduced by $2/\sigma_{ii}^2$.

12. A Run Test of the Hypothesis that the Median of a Stochastic Process is Constant. T. S. FERGUSON and CHARLES H. KRAFT, University of California.

Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of observations made at times $t_1 < t_2 < \cdots < t_n < \cdots$ on independent variables having a constant median equal to m. Let $Y_i = 1$ if $X_i > m$ and zero otherwise. Let r_k^+ (resp. r_k^-) be the number of runs of 1's (resp. 0's) of length k out of the first r runs. The joint distribution of $r_k^\pm$ $k = 1, 2, \cdots$ is found and the χ^2 test of the null hypothesis is proposed. Under alternative hypotheses for which $P\{Y_i = 1\}$ is periodic, the asymptotic distribution as $r \to \infty$ of $r_k^\pm$ is derived and the power of the test is found. Models are discussed for which the observations X_n form a stationary process with $E(X_nX_{n+k}) = \sigma^2\rho^{-k}$ and asymptotic properties of proposed tests of the analogous hypotheses are discussed.

13. Analysis of Dispersion on a Sphere. (By Title.) G. S. WATSON, The Australian National University.

In palaeomagnetism, the observations are directions of remanent magnetism of rock specimens. These observations may be regarded as position vectors of points on a unit sphere. The probability density, $\exp(k \cos \theta)$, where θ is the angle between the polar and observation vectors, has recently been suggested by Fisher for the analysis of such data. In this paper some exact tests and a full range of approximate tests, sufficiently accurate for most practical situations, are given for testing hypotheses about the precision constant k and the polar vector. The approximate tests, which involve only the F-distribution, arise from the fact that, when k is large, the points have a circular normal distribution about the pole.

14. Convergence in Distribution and Fourier-Stieltjes Transforms of Random Functions. (By Title. Preliminary Report) EMANUEL PARZEN, Columbia University.

Let $Y(t)$ and $Y_n(t)$, for $n = 1, 2, \cdots$, be random functions on $0 \leqq t \leqq 1$, whose sample functions belong to an (incomplete) Banach space $\mathfrak{B}$, usually the space of functions continuous except for a finite number of finite jumps. Define $Y_n(t)$ to converge in distribution to $Y(t)$ if, for every functional $g[y(t)]$ continuous in norm except for a set of functions of measure 0 according to the measure induced by $Y(t)$ on $\mathfrak{B}$, the random variable $g[Y_n(t)]$ converges in distribution to $g[Y(t)]$. Under this definition, it is immediately seen that if $T[y(t)]$ is an operator on $\mathfrak{B}$ to itself which is "continuous almost everywhere" as above, then $T[Y_n(t)]$ converges in distribution to $T[Y(t)]$. A general theorem, stating sufficient conditions for convergence in distribution and involving countable decompositions of the random functions, is given. It is applied to obtain quick proofs of various theorems of Donsker, involving consecutive sums of independent random variables and empirical distribution functions, but, up to the present time, only for the case of L_2 norm. In these applications, an important role is played by the Fourier-Stieltjes transform $Z_n(v) = \int_0^1 e^{2\pi ivt}\, dY_n(t)$, defined for $v = 0, \pm1, \pm2, \cdots$.

15. On the Statistical Analysis of Markov Chains. (By Title. Preliminary Report) LEO A. GOODMAN, University of Chicago.

T. W. Anderson has studied statistical inference in Markov chains with particular application to data in which each observation is a sequence of states, over a finite number T of time points, from a Markov chain with the same transition probability matrices $P_t = \{p_{ij}(t-1, t)\}$, and $n_i(0)$ observations are in state i at the time origin (*Ann. Math. Stat.*, Vol. 22 (1951), p. 607). He assumes that $n_i(0) \to \infty$, and presents likelihood ratio tests for the following hypotheses: (a) P_t is stationary (i.e., $P_t = P = \{p_{ij}\}$) against alternatives that it varies over time; (b) P is a given matrix (or that certain elements of P are given); and (c) the process is first order against the alternative that it is second order. The present paper presents χ^2 tests of goodness of fit which are analogous to the likelihood ratio tests for hypotheses (a), (b), and (c), and similar tests are also presented for the following hypotheses: (c') that the process is rth order against the alternative that it is $r+1$th order; and (d) that s samples of observations are samples from the same Markov chain P. P. G. Hoel (*Biometrika*, Vol. 41 (1954), p. 430) has presented a likelihood ratio test of (c') when a single observation (sequence of states) is obtained and $T \to \infty$ in the 'positively regular' case. The present paper presents an analogous χ^2 test of goodness of fit for this case. An advantage of the χ^2 tests of goodness of fit presented in the present paper is that, for many users of these results, the motivation for these tests and the application of the methods presented is simpler.

16. Convergence Properties of Elements of a Certain Class of Stochastic Approximation Processes, Continued. (By Title.) DONALD L. BURKHOLDER, University of North Carolina.

For each natural number n and each real number x let $Z_n(x)$ be a random variable such that $R_n(x) = E\, Z_n(x)$ exists. Suppose θ is a real number, γ is a positive number and $\{\mu_n\}$ is a real number sequence such that $(x - \mu_n)R_n(x) > 0$, for each pair n, x where $x \neq \mu_n$, and $\theta - \mu_n = 0(n^{-\gamma})$ as $n \to \infty$. Then there exists a stochastic approximation process $\{x_n\}$ such that, under several different sets of further conditions, $n^{\xi}(x_n - \theta)$ is asymptotically normal where $0 < \xi \leqq \frac{1}{2}$, $\xi < \gamma$. The theorems proved for the above case contain Chung's Theorem 9, *Ann. Math. Stat.*, Vol. 25 (1954), pp. 463–483, on the class of Robbins-Monro processes. The theorems also imply asymptotic normality of the Kiefer-Wolfowitz processes under fairly general conditions, and asymptotic normality of certain stochastic approximation processes useful in estimating the mode of a density function, also, under fairly general conditions.

(Abstracts of papers presented at the Ann Arbor meeting of the Institute, August 30–September 2, 1955)

1. Distribution of the m-th Range, J. ARTHUR GREENWOOD, Manhattan Life Insurance Company.

Gumbel [*Ann. Math. Stat.*, vol. 18 (1947), p. 410] has expressed the distribution of the m-th range as the convolution of the distribution of m-th extremes. The result of Garti and Consoli [*Studies presented to R. v. Mises*, New York, 1954, p. 302, equation (1)] is applied to express the differential of probability of the m-th range in terms of Bessel functions of the third kind. Integration by parts yields the distribution.

2. Approximation to the Distribution of the Sum of Cosines of Random Angles (Preliminary Report), DAVID DURAND, Massachusetts Institute of Technology, and J. ARTHUR GREENWOOD, Manhattan Life Insurance Company.

It has long been known that each component of a circularly symmetric random walk of n steps is normally distributed with error $O(1/n)$. The authors recently found (*Ann. Math. Stat.*, Vol. 26 (1955), p. 237) that $V = \Sigma \cos \xi_\nu$, where the angles ξ_ν are independently and uniformly distributed, has the characteristic function $[J_0(t)]^n$. Numerical values of the cumulants are found by expanding the log of the characteristic function, and the normal approximation is improved by the expansion $Pr[V \leqq z(n/2)^{\frac{1}{2}}] = \Phi(z) - n^{-1}(\phi^{iii}/16) + n^{-2}(\phi^{v}/72 + \phi^{vii}/512) - n^{-3}(11\phi^{vii}/3072 + \phi^{ix}/1152 + \phi^{xi}/24576) + n^{-4}(19\phi^{ix}/19200 + 425\phi^{xi}/1327104 + \phi^{xiii}/36864 + \phi^{xv}/1572864) + o(n^{-4})$, where $\phi(z)$ is the standard normal density, ϕ^{j} is its jth derivative, and Φ is its integral with lower limit $-\infty$. The expansion is inverted for use in computing percentage points.

3. Crystalline Connectivity, J. M. HAMMERSLEY, University of Oxford.

The paper studies the number of self-avoiding walks on an abstract formulation of the connective structure of atomic crystals. The theory bears on the behaviour of absorbent media.

4. Distributions of Roots of Algebraic Equations with Variable Coefficients, JOHN W. HAMBLEN, Oklahoma Agricultural and Mechanical College.

Consider an algebraic equation which can be written in polynomial form as (1) $\eta^n - \xi_1\eta^{n-1} + \xi_2\eta^{n-2} - \cdots + (-1)^n\xi_n = 0$, where the coefficients, ξ_i $(i = 1, \cdots, n)$, are real or complex random variables with a given joint p.d.f. The roots of (1), η_i ($i = 1, \cdots, n$) are random variables which have a p.d.f. that depends upon the p.d.f. of the coefficients. The problem under consideration is to determine the p.d.f. of the roots given the p.d.f. of the coefficients. The case where the ξ_i are complex random variables was considered in a note by M. A. Girshick (*nAnals of Mathematical Statistics*, Vol. 13, (1942), p. 235.). When the ξ_i are real, the η_i may be real or complex. For real η_i the functional form of their p.d.f. is obtained by a change of variables in the p.d.f. of the ξ_i by the use of the relationships (2) $\xi_1 = \Sigma_{i=1}^{n} \eta_i$, $\xi_2 = \Sigma_{i<j} \eta_i \cdot \eta_j, \cdots, \xi_n = \Pi_{i=1}^{n} \eta_i$, with Jacobian, J, given by $\Pi_{i=j} (\eta_i - \eta_j)$. However, to define the p.d.f. of the η_i completely, the region of the root-space over which the p.d.f. is greater than zero must be determined. This region is, of course, dependent upon the subregions of the region in the coefficient-space for which the p.d.f. of the ξ_i is greater than zero which give real roots. For complex η_i the treatment is similar, but a new set of relationships must be found to replace (2). In this case, the ξ_i must be expressed as functions of the real and imaginary parts of the η_i separately.

5. On a Modified Sum of Poisson Processes, A. BRUCE CLARKE, University of Michigan.

Let l and m be positive integers and, for each i with $-l \leqq i \leqq m$, let $\mathrm{m}_i(t)$ be an integer-valued Poisson process with parameter λ_i, proceeding by steps of magnitude i, the processes being mutually independent with $\mathrm{m}_i(0) = 0$. Let $\mathrm{m}(t) = \Sigma_{i=-l}^{m} \mathrm{m}_i(t)$. $\mathrm{m}(t)$ will then be a temporally homogeneous process of independent increments with generating function $\phi(z, t) = E(z^{\mathrm{m}(t)}) = \exp [t\Sigma_{i=-l}^{m} \lambda_i(z^i - 1)]$, from which the probabilities $q_n(t) = P_r(\mathrm{m}(t) = n)$, $-\infty < n < \infty$, can be obtained. Let $\mathbf{n}(t)$ be a process having the same transition probabilities as $\mathrm{m}(t)$ except that the process is restricted to *nonnegative* integer values, transitions giving negative values being forbidden, e.g., if $l = m = 1$ this will be a simple one-step queuing process. Let $p_n(t)$, $0 \leqq n < \infty$, be the probability distribution of $\mathbf{n}(t)$, $p_n(t) = Pr(\mathbf{n}(t) = n)$. Using the Kolmogorov equations it can be shown that there exists a linear transformation giving the $p_n(t)$ in terms of the $q_n(t)$, $p_n(t) = \Sigma_{i=-\infty}^{\infty} a_{ik} q_k(t)$. In the case $m \geqq l$, this equation reduces to a convolution of the form $p_n(t) = \Sigma_{i=0}^{\infty} a_i q_{n+i}(t)$. Under certain conditions the coefficients a_i may be determined explicitly.

6. A Method of Constructing Nonparametric Multivariate Tests, (Preliminary Report), T. W. ANDERSON, Columbia University.

Let $x_1, \cdots, x_m$ be observations from $F(x)$ and $y_1, \cdots, y_m$ from $G(x)$. When the observations are scalar, nonparametric tests of the hypothesis $F(x) \equiv G(x)$ are based on the ranks of the observations or, equivalently, on $m_0, \cdots, m_n$, where m_i is the number of y's falling between the ith and $(i + 1)$-st ordered x's. When the observations are vectors, let $R_0, \cdots, R_n$ be $n + 1$ statistically equivalent blocks as defined by J. W. Tukey. ("Nonparametric estimation II. Statistically equivalent blocks and tolerance regions the continuous case," *Annals of Math. Stat.*, Vol. 18 (1947), pp. 529–539); let m_i be the number of y's falling in R_i. Under the null hypothesis the distribution of $m_0, \cdots, m_n$ is the same in the vector case as in the scalar case. The distribution of any test criterion based on the m's is the same in both the scalar and vector cases. The choice of the functions used to define the blocks and the subsequent test will depend on the relevant alternative hypotheses. These blocks for one sample can also be used to test the hypothesis that $F(x)$ is a specified distribution. (Work sponsored by School of Aviation Medicine, Randolph Field, Texas, under Contract A F 18 (600)–941.)

7. Confidence Intervals for a Measure of Effectiveness (Preliminary Report), GOTTFRIED E. NOETHER, Boston University.

Let p_1 and p_2 be the probabilities of success of two "treatments" and define $p = (p_2 - p_1)/(1 - p_1)$. For $p_1 < p_2$, p may be considered a measure of the greater effectiveness of treatment 2. Since $1 - p = q_2/q_1 = q$, say, the problem of finding a confidence interval for p is equivalent to that of finding a confidence interval for the ratio of the parameters of two binomial populations. If y_i, $i = 1, 2$, denotes the observed relative frequency of failure for the ith treatment, various confidence intervals for q based on normal approximations to the distributions of y_2/y_1 and $y_2 - qy_1$ are derived and compared. Confidence intervals based on joint confidence regions for q_1 and q_2 are also considered.

8. Truncated Binomial and Negative Binomial Distributions, Paul R. Rider, Wright-Patterson Air Force Base.

This paper derives a simple estimator of the parameter of a binomial distribution from which one or more classes have been truncated, also easily calculated estimators of the two parameters of a negative binomial distribution from which the zero class is missing. These estimators are analogous to that previously proposed by the author for the parameter of a truncated Poisson distribution (*Journal of the American Statistical Association*, Vol. 48 (1953), pp. 826–830). They compare favorably with maximum likelihood estimators. Illustrative examples are provided.

9. Error Rates and Sample Sizes for Multiple Range Tests, H. Leon Harter, Wright-Patterson Air Force Base.

A study is made of error rates and sample sizes for three multiple range tests (the Newman-Keuls test, Tukey's X procedure, and Duncan's New Multiple Range Test). Multiple range tests are used for testing the significance of the range of p out of m ordered means of samples of size N, where $p = 2, 3, \cdots, m$. For various combinations of m and N, Table 1 gives maximum and minimum Type I error rates α (as defined for these tests) when α_L (as defined for the LSD test) has the values 0.05 and 0.01. For various combinations of m and N, Table 2 gives maximum and minimum Type II error rates β for each of the multiple range tests as a function of $\delta = | \mu_U - \mu_L |/\sigma$, where μ_U and μ_L are the population means corresponding respectively to the largest and smallest of p sample means and σ is the population standard deviation. For various combinations of α, β and δ, Table 3 gives maximum and minimum required sample sizes N for each of the multiple range tests. Since, for each test, the critical range of p means is a non-decreasing function of p, the extreme values of α, β and N occur for $p = 2$ and $p = m$.

10. On Transient Markov Chains with a Countable Number of States, David Blackwell, University of California.

Let $X_1, X_2, \cdots$ form a Markov process with stationary transition probabilities and states the non-negative integers. A set I of states is called *almost closed* if Prob $\{X_n \varepsilon I$ infinitely often$\}$ = Prob $\{X_n \varepsilon I$ for all sufficiently large $n\} > 0$. It is shown that there is a decomposition of the set of states into disjoint almost closed sets $I_1, I_2, \cdots$ such that (a) all I_j except at most one are *atomic*, i.e. do not contain two disjoint almost closed subsets, (b) the non-atomic I_j, if present, has no atomic subsets, and (c) Σ_j Prob $\{X_n \varepsilon I_j$ infinitely often$\} = 1$. If there are independent identically distributed $Y_1, Y_2, \cdots$ with $X_n = Y_1 + \cdots + Y_n$, then the set of all states is atomic. The results are new only if the process has transient components. The main tool is the martingale convergence theorem.

11. A z-transformation and a t-statistic for Serial Correlation Coefficients, John S. White, University of Manitoba.

An approximate distribution for the sample serial correlation coefficient from a circularly correlated population has been obtained by R. B. Leipnik [*Annals of Mathematical Statistics*, Vol. 18 (1947) p. 80]. In this paper several aspects of Leipnik's distribution will be considered. The moments of the distribution are expressible in terms of hypergeometric functions which may be explicitly evaluated for specific values. The transformation, analogous to Fisher's z-transformation for product moment correlation coefficients, is found to be $z = \text{arc sin } r$. It is shown that z is asymptotically normally distributed with mean arc sin ρ, where ρ is the population parameter, and variance $1/T$. The statistic $t^* = \sqrt{N+1}\,(r - \rho)/\sqrt{1 - r^2}$ has a density function of the type $f(t^*) = s_{N+1}(t^*) + g(t^*)$, where $s_{N+1}(t^*)$ is a density function for "Student's t" with $N + 1$ degrees of freedom and $g(t^*)$ is an odd function. The function $g(t^*)$ goes to zero as the sample size increases. The statistic $(t^*)^2$ has an F distribution with 1 and $N + 1$ degrees of freedom. Examples are given for the use of z and t^* in testing hypotheses and in the construction of confidence intervals.

12. Distance Tests with Good Power for the Nonparametric k-sample Problem, J. Kiefer, Cornell University.

Let X_{ij} be independent $(1 \leqq i \leqq n_j, 1 \leqq j \leqq k)$, X_{ij} having unknown continuous distribution function (d.f.) F_j. For the nonparametric k-sample problem $(k \geqq 2)$ of testing $H: F_1 = F_2 = \cdots = F_k$, most known tests have only been shown to have desirable consistency or power properties against limited classes of alternatives. Let $S_j(x)$ be the sample d.f. of the n_j observations in the jth set. A wide variety of tests of H may be based on various measures of distance (dispersion) among the S_j which generalize the (possibly weighted) Kolmogorov-Smirnov and ω^2-type criteria for $k = 2$. For example, supposing all $n_j = n$, possible critical regions are large values of $V_n = \sup_{q,r,x} | S_q(x) - S_r(x) |$ or of $T_n = \sup_x \Sigma_j (S_j(x) - \bar{S}(x))^2$ or of $\int \Sigma_j(S_j(x) - \bar{S}(x))^2 \, d\bar{S}(x)$, where $\bar{S}(x) = \Sigma_j S_j(x)/k$. Criteria like V_n and T_n will usually have the better power properties: e.g., for $0 < \alpha, \beta < 1$, there exists a value $\delta(\alpha, \beta)$ such that the test of size α based on V_n or T_n has power $> \beta$ against all alternatives for which $\sup_{q,r,x} | F_q(x) - F_r(x) | > \delta(\alpha, \beta)/\sqrt{n}$. Limiting d.f.'s under H may be found by several methods: e.g., that of nT_n may be obtained by finding in $k - 1$ dimensions the probability of absorption of a Wiener process by a sphere whose radius is $b(1 + t)$ at time t. Other tests using only previously known limiting d.f.'s also have good power properties. Analogous tests may be used for testing $H': F_1 = \cdots = F_k = G$ where G is specified.

13. Comparison of Populations Whose Growth Can Be Described by a Branching Stochastic Process, A. T. Bharucha-Reid, Columbia University.

In this paper the sequential procedure of Girshick [*Ann. Math. Stat.*, Vol. 17, (1946), pp. 123–143] for comparing or ranking two populations is applied to some populations whose growth can be described by a branching stochastic process (b.s.p.). Let Λ_1 and Λ_2 be two populations, each of whose growth or development can be represented by the b.s.p. $\{X(t; \omega), t \geqq 0\}$. The probability density $p(t, x; \omega) = P(X(t; \omega) = x)$, $x = 0, 1, \cdots$, is assumed to be known except for the value of the parameter ω. It is of interest to test the composite hypothesis $H_1: \omega_1 < \omega_2$ against the alternative hypothesis $H_2: \omega_1 > \omega_2$, where ω_i is the value of the parameter in $p(t, x_i; \omega_i)$, $i = 1, 2$. A Wald sequential probability ratio test is set up as follows: Select two constants A and B, $B < 0 < A$, and two values of ω, say $\hat{\omega}_1$ and $\hat{\omega}_2$, where $\hat{\omega}_1 < \hat{\omega}_2$. The two populations are observed continuously, and on the basis of the realizations $x_1(t)$ and $x_2(t)$ the decision function $d(t) = \log \{p(t, x_1; \hat{\omega}_2)p(t, x_2; \hat{\omega}_1)/p(t, x_1; \hat{\omega}_1)p(t, x_2; \hat{\omega}_2)\}$ is computed. If for any $t = T$ $d(T) \leqq B$ we accept H_2. If $d(T) \geqq A$ we accept H_1. If neither holds we continue to observe. For those b.s.p. which admit a sufficient statistic for the parameter ω, a new decision function, depending on the realizations alone, can be defined. In these cases the decision boundaries are functions of time and the parameter values $\hat{\omega}_1$ and $\hat{\omega}_2$. The test is applied to the birth, death, birth-and-death, and Pólya processes. Possible applications of this procedure to some problems in biology, physics, sociology, and telephone and industrial engineering are discussed. A detailed treatment of a problem concerned with the comparison of two stochastic epidemics is given.