ON THE RATIO OF VARIANCES IN THE MIXED INCOMPLETE BLOCK MODEL[1]

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Summary. The present article is an extension of Wald's paper, [12] "A note on regression analysis." Confidence interval and testing procedures on a ratio of variances are given for continuously more specific models with a corresponding increase in the preciseness of the results. Finally, Linked Block Designs and Duals of Partially Balanced Designs with two associative classes are discussed and it is found that here the analysis is quite simple indeed. (Ordinary Lattices belong to this last group.)

1. Introduction. Though the experimenter has been using the variance components model for nearly as long as the fixed effects model, there has been relatively little success in developing a complete theory such as the least squares approach to the fixed effect case. Among the few theoretical papers on this subject is a series due to Abraham Wald which culminates in his 1947 paper [12]. There he outlines a method of placing a confidence interval on a ratio of variances. The actual application of this method in the nonorthogonal case will depend on the solution of an equation of the n-th degree where n would in general be large. The question naturally arises to what kind of designs can Wald's method be applied in practice without unduly complicated calculations. One object of this paper is to answer this question so far as it relates to incomplete block designs.

In Section 2 a set of sufficient statistics is derived for the variance components model with errors arising from only two sources. These statistics are then used in Section 3 to derive confidence intervals and tests of hypotheses concerning the ratio μ of the two components of variance. In Section 4 these results are applied to incomplete designs. It is shown that if we have a design with linked blocks, i.e., any two blocks have the same number of treatments in common, then the formulae for finding the confidence interval take a very simple form.

It appears that for carrying out tests simply or for assigning confidence intervals to the ratio of the per plot error to the block error one must balance the design with respect to the blocks, just as a balance with respect to the treatments enables one to estimate with ease, and carry out tests concerning 'treatment effects'. This line of thought has been pursued by investigating 'partially

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linked' designs, for which the configuration of the blocks obeys the same restrictions, as does the configuration of treatments for a 'partially balanced' design. It has been shown that the matrix of the least square equations which would arise in estimating the block effects when regarded as fixed, has not more than m^* distinct non-zero characteristic roots, if there are m^* associate classes (with respect to the blocks). In the important practical case when $m^* = 2$, there are two distinct characteristic roots, which are roots of the quadratic $e^2 - He + \Delta = 0$ where H and Δ are the same constants which appear in the interblock analysis of the dual of the design. Since these constants have already been tabulated for all known 2 associative class designs [3], the calculation of the roots e' and e'' is very simple. The confidence interval and Wald's test depend only on these roots, and the actual sums of squares involved can be simply calculated. Results regarding the number of distinct characteristic roots have been very recently obtained by Connor and Clatworthy [5] in an entirely different connection (viz., the combinatorial properties of balanced incomplete block designs). It thus appears that designs which are completely or partially balanced with respect to treatments, and completely or partially linked with respect to blocks are of special importance.

2. The least squares model with errors arising from two sources.

2.1 Motivation. The purpose of this section is to derive a set of sufficient statistics for the "mixed" variance components model with errors arising from only two sources. We do this by treating our observations as random variables whose conditional distribution is the same as that assumed unconditionally in the ordinary least squares problem. This approach is that of Wald [12].

2.2 Notation. We shall be using vector and matrix notation throughout the rest of this paper. Small Roman or Greek letters will denote vectors or scalors while the capital Roman letters will be reserved for matrices. All vectors will be column vectors unless they are primed, in which case they will be row vectors. $X = X(N \times b + u)$ will mean that the matrix X has N rows and $b + u$ columns.

$A = \begin{bmatrix} A_1, & A_2 \\ A_3, & A_4 \end{bmatrix}$ will mean that the matrix A has been partitioned into the four matrices A_1, A_2, A_3, A_4 and that these last four matrices have the positions indicated.

2.3 A theorem in least squares. Let $y_1, y_2, \cdots, y_N$ be independent random variables with a common variance σ^2 and let

$$2.3.1 \qquad y' = (y_1, y_2, \cdots, y_N).$$

Suppose in addition that

$$2.3.2 \qquad E(y) = X\beta = (X_1, X_2)\begin{bmatrix} \beta_{(1)} \\ \beta_{(2)} \end{bmatrix}$$

where

$$X = X(N \times b + u) = (X_1(N \times u), X_2(N \times b)) = (X_1, X_2)$$

is a matrix of known constants and β is a column vector whose elements are the unknown parameters $\beta_1, \beta_2, \cdots, \beta_u, \beta_{u+1}, \beta_{u+2}, \cdots, \beta_{b+u}$. We also introduce the notation

2.3.3 $$A = A(b + u \times b + u) = \begin{bmatrix} A_1, A_2' \\ A_2, A_3 \end{bmatrix}$$

where

$$A_1 = X_1'X_1, \qquad A_2 = X_2'X_1, \qquad A_3 = X_2'X_2.$$

It is a well known theorem in least squares that if g^* is a linear function of the observations, and if g^* estimates a linear function of $\beta_{u+1}, \beta_{u+2}, \cdots, \beta_{b+u}$ then g^* must be a linear function of the elements of

2.3.4 $$p = p(b \times 1) = (X_2' - A_2A_1^{-1}X_1')y.$$

Note also $E(p) = D\beta_{(2)}$, where D is equal to $A_3 - A_2A_1^{-1}A_2'$, and from the formulae for the variances and covariances of linear functions we see that the variance-covariance matrix of the p's is:

2.3.5 $$(X_2' - A_2A_1^{-1}X_1')(X_2' - A_2A_1^{-1}X_1')'\sigma^2 = D\sigma^2.$$

We define $g_1 = g_1(u \times 1) = X_1'y$. The elements of the vector g_1 generate a vector space V_1 of linear functions. This space has dimensionality u since we have assumed that $A_1(= X_1'X_1)$ has an inverse. The elements of p also generate a vector space of linear functions. We denote this space by V_2 and its dimensionality by r, say. A short calculation shows that the bases of V_1 and V_2 are orthogonal and hence the spaces are orthogonal. We now define the space V to be that orthogonal to V_1 and V_2. The dimensionality of V then must be $N - u - r$. We may now choose an orthogonal basis for V, say $Y_1, \cdots, Y_{N-u-r}$; it is easily proved that $E(Y_j) = 0$. We remark for future reference that $\sum Y_i^2$ is the sum of squares due to error.

We record

2.3.6 $$\begin{aligned} E(g_1) &= A_1\beta_{(1)} + A_2'\beta_{(2)}, \\ E(p) &= D\beta_{(2)}, \\ E(Y_i) &= 0. \end{aligned}$$

And the covariance matrix of the elements of Y^* where

$$Y^{*\prime} = (g_1', p', Y_1', \cdots, Y_{N-u-r})$$

is

2.3.7 $$\begin{bmatrix} A_1 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & I \end{bmatrix} \sigma^2.$$

2.4 Variance components and conditional random variables. We will now change our assumptions somewhat from Section 3. Suppose now that $y_1, \cdots, y_N$ are independent and normal for given $\beta_{(2)}$ with means $E(y/\beta_{(2)}) = X\beta$ and variances σ^2. In addition suppose that the elements of $\beta_{(2)}$ are independent and identically normal with means 0 and variances ζ^2. We see from 2.3.6 and 2.3.7 that

2.4.1
$$\begin{aligned} E(g_1/\beta_{(2)}) &= A_1\beta_{(1)} + A_2'\beta_{(2)}\,. \\ E(p/\beta_{(2)}) &= D\beta_{(2)} \\ E(Y_i/\beta_{(2)}) &= 0. \end{aligned}$$

And the covariance matrix of these conditional random variables is

2.4.2
$$\begin{bmatrix} A_1 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & I \end{bmatrix} \sigma^2.$$

We now state several lemmas, the last two of which were independently developed by Madow [9] and Skibinsky [10].

2.4.3
$$\begin{aligned} &1)\ E\,E(x/z) &&= E(x) \\ &2)\ \text{Var}\,(x) &&= E\,\text{Var}\,(x/z) + \text{Var}\,E(x/z) \\ &3)\ \text{Cov}\,(x, y) &&= E\,\text{Cov}\,(x, y/z) + \text{Cov}\,[E(x/z), E(y/z)]. \end{aligned}$$

Applying these lemmas we find the unconditional means to be

2.4.4
$$E(g_1) = A_1\beta_{(1)}\,, \qquad E(p) = 0, \qquad E(Y_j) = 0.$$

The unconditional covariance matrix of these same variables is

2.4.5
$$\begin{bmatrix} A_1 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & I \end{bmatrix} \sigma^2 + \begin{bmatrix} A_2'A_2 & A_2'D & 0 \\ (A_2'D)' & D^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \zeta^2$$

where $D = (A_3 - A_2A_1^{-1}A_2')$.

We may make an orthogonal transformation, $p = Mz$, in a manner entirely analogous to that of [11] Section 1 and find that $z_1, \cdots, z_r$ and $\sum Y_j^2$ are a set of joint sufficient statistics for the parameters $\beta_1, \cdots, \beta_u, \sigma^2$ and ζ^2.

3. The Ratio ζ^2/σ^2.

3.1 Motivation. In this section we continue assuming the variance components model discussed in Section 2. In that section we found a set of statistics to which we may restrict ourselves in inference problems concerning all the parameters of the distribution. Using those statistics we will consider confidence interval and hypothesis testing problems involving the ratio $\zeta^2/\sigma^2 = \mu$, say.

3.2 The Wald confidence interval [12] *for μ and an associated test.* We may verify that

3.2.1
$$F = F(\mu) = \frac{n}{r}\sum \frac{z_i^2}{e_i + e_i^2\mu} \Big/ \sum Y_j^2$$

actually has the F distribution with r and n degrees of freedom. This together with the fact that $F(\mu)$ is a monotone decreasing function enables Wald to find a confidence interval for μ. We summarize his result as:

THEOREM 1. *If F_1 and F_2 are chosen so that* $\Pr(F_1 \leqq F \leqq F_2) = 1 - \alpha$, *and if μ_s is the largest root in μ of $F(\mu) = F_s$, $s = (1, 2)$ where $F(\mu)$ is given by* 3.2.1, *then (μ_2, μ_1) is a $1 - \alpha$ confidence interval for μ if μ_2 is positive and $(0, \mu_1)$ is a $1 - \alpha$ confidence interval if μ_2 is negative. If μ_1 is negative then $(0, 0)$ is a degenerate $1 - \alpha$ confidence interval.*

Of course, this confidence interval is not unique, as many possible choices of F_1 and F_2 are possible. For example, if $F_2 = \infty$ then $\mu_2 = 0$ and Wald's procedure gives an upper confidence limit; if $F_1 = 0$ then $\mu_1 = \infty$ and we have a lower confidence bound for μ.

We may also derive a test of $H_0: \mu \leqq \mu_0$ vs. $H_1: \mu > \mu_0$ in this manner.

THEOREM 2. *If $F(\mu_0)$ is given by* 3.2.1 *and c is determined appropriately, if we are testing $H_0: \mu \leqq \mu_0$ vs. $H_1: \mu \geqq \mu_0$, and if we accept H_0 when $F(\mu_0) < c$ and otherwise reject H_0; then the power function of this test is an increasing function of μ.*

The power function is as follows:

$$\beta(\mu) = \text{const.} \int_{F(\mu_0)>c} \exp\left[-\frac{1}{2}\left(\frac{\sum Y_j^2}{\sigma^2}\right) + \sum \frac{z_i^2}{e_i\sigma^2 + e_i^2\zeta^2}\right)\right] dY\, dz$$

$$= \text{const.} \int_{G(\mu)>c} \exp\left[-\tfrac{1}{2}\left(\sum f_j^2 + \sum g_i^2\right)\right] df\, dg,$$

where

$$G(\mu) = \left(\frac{n}{r}\right) \sum \frac{1 + e_i\mu}{1 + e_i\mu_0}\, g_i^2 \Big/ \sum f_j^2,$$

and $g_i = z_i/(e_i\sigma^2 + e_i^2\zeta^2)^{1/2}$; $i = 1, \cdots, r$, and $f_j = Y_j/\sigma$; $j = 1, \cdots, n$. Thus $G(\mu)$ is an increasing function of μ. Also if $G_1(\mu) \leqq G_2(\mu)$ then $c < G_1(\mu)$ implies that $c < G_2(\mu)$ so that

$$\int_{G_1(\mu)>c} dF \leqq \int_{G_2(\mu)>c} dF$$

where $dF = \text{const.} \exp\left[-\tfrac{1}{2}\left(\sum f_j^2 + \sum g_i^2\right)\right] df\, dg$. Therefore β is an increasing function of G and thus of μ.

Thus if we choose c so that $\beta(\mu_0) = \alpha$ then we have an α level test which has appropriate power properties. These appropriate power properties are, of course, that we are more likely to say $\mu < \mu_0$ the smaller μ is and we are less likely to say that $\mu < \mu_0$ the larger μ is. $\beta(\mu_0)$ is the probability that a statistic with the F distribution exceeds a constant c. Hence c is chosen so that $F_{r,n}(c) = \alpha$ where $F_{r,n}$ is the cumulative F distribution with r degrees of freedom in the numerator and n in the denominator. Note that when μ_0 is 0 that

$$F(\mu_0) = \frac{n}{r} \sum \frac{z_i^2}{e_i} \Big/ \sum Y_j^2.$$

4. Some properties of incomplete block designs.

4.1 Motivation. In 4.2 we will see what some of the formulae and theory of 2.3 and 2.4 say when specialized to incomplete block designs. Then in 4.3 and 4.4 we will consider the computation of

$$\sum \frac{z_i^2}{e_i + e_i^2 \mu}$$

in some particularly interesting special cases. It will be remembered that this is the quantity on which the test and the confidence interval procedures of Section 3 are based.

4.2 Application to General Incomplete Block Designs. We now consider $y_{ij}(i = 1, \cdots, u; j = 1, \cdots, b)$ to be the "yield" from the i^{th} "treatment" and j^{th} "block" of a statistical experiment using an incomplete block design. The reason for the quotes above is to remind the reader that these terms may apply to applications which are not at all agricultural in nature. We further assume that the y_{ij}'s are independent and normally distributed random variables for given block effects $\alpha_1, \cdots, \alpha_b$ with means: $E(y_{ij}/\alpha_1, \cdots, \alpha_b) = n_{ij}(\tau_i + \alpha_j)$ and variance σ^2. Here n_{ij} is 1 or 0 according as the i^{th} treatment does or does not occur in the j^{th} block. Thus τ_i is the i^{th} treatment effect. Since only those y_{ij} are considered for which $n_{ij} = 1$, the total number of observations y_{ij} is N (i.e. $\sum_{ij} n_{ij} = N$). In addition the α's are independent and identically normally distributed with mean 0 and variance ζ^2. Note that if the α's were unknown parameters instead of random variables then we would have the general incomplete block model with fixed effects which appears in analysis of variance (see for example Bose [1]).

We may see that this model is a special case of the one described in 2.4 by setting

$$\text{4.2.1} \qquad X = \begin{bmatrix} n_{11} & 0 & \cdots & 0 & n_{11} & 0 & \cdots & 0 \\ n_{12} & 0 & \cdots & 0 & 0 & n_{12} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ n_{1b} & 0 & \cdots & 0, & 0 & 0 & \cdots & n_{1b} \\ 0 & n_{21} & \cdots & 0 & n_{21} & 0 & \cdots & 0 \\ 0 & n_{22} & \cdots & 0 & 0 & n_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & n_{2b} & \cdots & 0, & 0 & 0 & \cdots & n_{2b} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & n_{u1} & n_{u1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & n_{u2} & 0 & n_{u2} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & n_{ub}, & 0 & 0 & \cdots & n_{ub} \end{bmatrix}$$

where if the i^{th} treatment does not occur in the j^{th} block, that is $n_{ij} = 0$, then it is understood that the corresponding row is missing from X. We must also let β be the column vector whose elements are $\tau_1, \tau_2, \cdots, \tau_u, \alpha_1, \alpha_2, \cdots,$

α_b. The r of Section 2 here becomes $b - 1$ and A_1 and A_3 become diagonal matrices with $\sum_j n_{ij}^2$ and $\sum_i n_{ij}^2$ respectively in the diagonal. It is easy to see that $\sum_j n_{ij}^2 = r_i$, the number of blocks in which the i^{th} treatment appears, and $\sum_i n_{ij}^2 = k_j$, the number of treatments in the j^{th} block. Thus

4.2.2
$$A_1 = \begin{bmatrix} r_1 & & & & 0 \\ & r_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & r_u \end{bmatrix}$$

and

4.2.3
$$A_3 = \begin{bmatrix} k_1 & & & & 0 \\ & k_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & k_b \end{bmatrix}$$

Also A_2 becomes

4.2.4
$$\begin{bmatrix} n_{11} & n_{21} & \cdots & n_{u1} \\ n_{12} & n_{22} & \cdots & n_{u2} \\ \vdots & \vdots & & \vdots \\ n_{1b} & n_{2b} & \cdots & n_{ub} \end{bmatrix}$$

and hence d_{ij}, the general element of $D = A_3 - A_2 A_1^{-1} A_2'$ is given by

4.2.5
$$d_{ij} = \delta_{ij} k_j - \sum_s \frac{n_{si} n_{sj}}{r_s}$$

where δ_{ij} is Kroneker's delta.

Let λ_{ij}^* be the number of treatments occurring both in the i^{th} and j^{th} blocks. Then if in particular $r_1 = r_2 = \cdots = r_u = r$ (say) we have

4.2.6
$$d_{ij} = \delta_{ij} k_j - \frac{\lambda_{ij}^*}{r}.$$

Remember from 2.3.4 that $p = (X_2' - A_2 A_1^{-1} X_1')y$; and hence if we have an incomplete block design we find that

4.2.7
$$p_i = B_i - \frac{n_{1i} T_1}{k_1} - \frac{n_{2i} T_2}{k_2} - \cdots - \frac{n_{ui} T_u}{k_u} \qquad i = 1, \cdots, b,$$

which are known as adjusted block totals. Here B_i is the total of the i^{th} block and T_j is the total of the j^{th} treatment.

4.3 Linked Block Designs. An incomplete block design has been defined to be a linked block design if

(a) Each block has the same number of treatments k,
(b) Each treatment occurs in r blocks,
(c) Any two blocks have the same number of treatments λ^* in common.
These designs were used by Youden [13], and are duals of the well-known balanced incomplete block designs. In this case our matrix D is of the form

4.3.1
$$\begin{bmatrix} k(1 - 1/r) & -\lambda^*/r & \cdots & -\lambda^*/r \\ -\lambda^*/r & k(1 - 1/r) & \cdots & -\lambda^*/r \\ \vdots & \vdots & & \vdots \\ -\lambda^*/r & -\lambda^*/r & \cdots & k(1 - 1/r) \end{bmatrix}$$

where λ^* is the number of times two blocks contain the same treatment. $|D - eI| = [1/r(k[r - 1] - \lambda^*) - e]^{b-1} \cdot (-e)$, since $k(r - 1) = (b - 1)\lambda^*$. Thus the characteristic roots of D are 0 and $e = 1/r(k[r - 1] - \lambda^*)$, the latter of multiplicity $b - 1$.

If now we make our orthogonal transformation of 2.4 from p's to z's, we find that $\sum z_i^2/(e_i + e_i^2\mu) = \sum z_i^2/(e + e^2\mu)$, since all the non zero characteristic roots are the same. However, since our transformation from p's to z's was orthogonal, $\sum p_i^2 = \sum z_i^2$, and

4.3.2
$$F(\mu) = \frac{N - u - b + 1}{b - 1} \frac{1}{e + e^2\mu} \frac{\sum p_i^2}{\sum Y_j^2}.$$

Now from 4.3.2, we know that μ_s is the solution in μ of

$$\frac{N - u - b + 1}{b - 1} \frac{1}{e + e^2\mu} \frac{\sum p_i^2}{\sum Y_j^2} = F_s,$$

where s is either 1 or 2. More explicitly

4.3.3
$$\mu_s = \left(\frac{1}{e^2 F_s} \frac{N - u - b + 1}{b - 1} \frac{\sum p_i^2}{\sum Y_j^2} - \frac{1}{e}\right) \qquad s = 1, 2.$$

Theorem 1 now supplies two-sided as well as one-sided confidence intervals for μ. Theorem 2 supplies a test of the hypothesis $H_0:\mu \leqq \mu_0$ vs. $H_1:\mu \geqq \mu_0$; here $r = b - 1$ and $n = N - u - b + 1$.

4.4 Partially Linked Designs. The dual of an incomplete block design is obtained by interchanging the rolls of the treatments and blocks. We now define a class of designs which are duals of the well known Partially Balanced Incomplete Block Designs. In analogy with Linked Block Designs we define the Partially Linked Designs to be those which satisfy the following conditions:

(i) The experimental material is divided into b blocks of k units each, different treatments being applied to the units in the same block.

(ii) There are u treatments, each of which occurs in r blocks.

(iii) There can be established a relation of association between any two blocks satisfying the following requirements:

a) Two blocks are either 1st, 2nd, $\cdots$, or $m^{*\text{th}}$ associates.

b) Each block has exactly n_i^*, i^{th} associates ($i = 1, 2, \cdots, m^*$).

c) Given any two blocks which are i^{th} associates, the number of blocks common to the j^{th} associates of the first, and the k^{th} associates of the second is p_{jk}^{i*} and is independent of the pair of blocks with which we start. Also $p_{jk}^{i*} = p_{kj}^{i*}$.

(iv) Two blocks which are i^{th} associates contain exactly λ_i^* common treatments. Hence for a partially linked incomplete block design

4.4.1 $$d_{ij} = \delta_{ij} k_j - \sum_s \frac{n_{si} n_{sj}}{r_s} = \begin{cases} k\left(1 - \dfrac{1}{r}\right) & \text{for } i = j \\ -\dfrac{\lambda_s^*}{r} & \text{for } i \neq j \end{cases}$$

where blocks i and j are s^{th} associates.

The following identities hold between the parameters of partially linked designs:

$$bk = ur, \qquad n_1^* + n_2^* + \cdots + n_{m^*}^* = b - 1,$$

$$n_1^*\lambda_1^* + n_2^*\lambda_2^* + \cdots + n_{m^*}^*\lambda_{m^*}^* = k(r - 1),$$

$$\sum_{k=1}^{m^*} p_{jk}^{i*} = \begin{cases} n_j^* & \text{for } i \neq j \\ n_j^* - 1 & \text{for } i = j, \end{cases}$$

$$n_i^* p_{jk}^{i*} = n_j^* p_{ik}^{j*} = n_k^* p_{ij}^{k*}.$$

The next theorem follows easily from the work of Connor and Clatworthy [5].

THEOREM 3. *If e is a non-zero characteristic root of $D = (d_{ij})$, then e is a root of the following determinental equation*

$$\begin{vmatrix} k(1 - 1/r) + \lambda_1^*/r - e & k(1 - 1/r) + \lambda_2^*/r - e \\ -\dfrac{p_{21}^{1*}}{r}(\lambda_1^* - \lambda_2^*) & k(1 - 1/r) + \dfrac{\lambda_2^*}{r} - \dfrac{p_{21}^{2*}}{r}(\lambda_1^* - \lambda_2^*) - e \end{vmatrix} = 0.$$

We denote the two roots of this equation by e' and e''. The method used by Connor and Clatworthy will also give the multiplicities ρ_1, ρ_2 of e', e'' in terms of e' and e''. We do not use the exact values of ρ_1, ρ_2 in this paper, but only the fact that they are positive integers which sum to $b - 1$.

We may verify that if H^* is the sum of the two characteristic roots and if Δ^* is their product, then

$$rH^* = (2kr - 2k + \lambda_1^* + \lambda_2^*) + (p_{12}^{1*} - p_{12}^{2*})(\lambda_1^* - \lambda_2^*)$$

and

4.4.2 $$\begin{aligned} r^2\Delta^* = {} & (kr - k + \lambda_1^*)(kr - k + \lambda_2^*) \\ & + (\lambda_1^* - \lambda_2^*)[k(r - 1)(p_{12}^{1*} - p_{12}^{2*}) + \lambda_2^* p_{12}^{1*} - \lambda_1^* p_{12}^{2*}]. \end{aligned}$$

If the dual of the design we are working with is tabulated in Bose, Clatworthy and Shrikhande [3] then H^* and Δ^* are the H and Δ tabulated there for this dual design.

We now evaluate the required sums of squares. Remember from 2.4 that $z = M'p$ where M is an orthogonal matrix such that $M'DM$ is a diagonal matrix with the characteristic roots of D in the diagonal. Thus if we deal with a partially linked design with two associative classes, then according to the results of this paragraph we may choose M so that

$$M'DM = \begin{bmatrix} e'I_{\rho_1}, & 0, & 0 \\ 0, & e''I_{\rho_2}, & 0 \\ 0, & 0, & 0 \end{bmatrix} = \begin{bmatrix} D_e, & 0 \\ 0, & 0 \end{bmatrix} = D^*,$$

say, where the subscript on the identity matrix indicates its dimensionality. In this case we find that

$$\sum \frac{z_i^2}{e_i + e_i^2 \mu}$$

is somewhat simplified. It is

4.4.4 $$\frac{\sum_1}{e' + (e')^2\mu} + \frac{\sum_2}{e'' + (e'')^2\mu}$$

where

$$\sum\nolimits_1 = \sum_{i=1}^{\rho_1} z_i^2 \quad \text{and} \quad \sum\nolimits_2 = \sum_{i=\rho_1+1}^{b-1} z_i^2 .$$

Now suppose we consider the quadratic form

4.4.5 $$m'(D - e'I)p$$

where m is a solution of

4.4.6 $$Dm = p.$$

We make the substitutions $p = Mz$ and $n = M'm$ or $m = Mn$.

$$Dm = Mz, \qquad DMn = Mz, \qquad M'DMn = z, \qquad D^*n = z.$$

$$z_i = \begin{cases} e'm_i\,; & m_i = z_i/e' \qquad i = 1, \cdots, \rho_1 \\ e''m_i\,; & m_i = z_i/e'' \quad i = \rho_1 + 1, \cdots, b - 1 \\ 0 & i = b. \end{cases}$$

Now using the above relationships it can be verified that

$$m'(D - e'I)p = \frac{e'' - e'}{e''} \sum\nolimits_2$$

so that $\sum_2 = M'(D - e'I)p\, e''/(e'' - e')$ but $m'(D - e'I)p = p'p - e'm'p$ and

4.4.7 $$\sum_2 = \frac{e''}{e'' - e'}(p'p - e'm'p).$$

Similarly,

$$\sum_1 = \frac{e'}{e' - e''}(p'p - e''m'p);$$

$m'p$ is called the block adjusted sum of squares and $p'p$ is the sum of squares of the adjusted block totals.

We may now calculate μ_1 and μ_2 for partially linked designs. To do so we must solve the equations:

4.4.8 $$F(\mu) = F_s \qquad s = 1, 2$$

or

4.4.9 $$\frac{\sum_1}{e' + (e')^2\mu} + \frac{\sum_2}{e'' + (e'')^2\mu} = a_s, \qquad s = 1, 2$$

where

4.4.10 $$a_s = F_s \sum Y_j^2 (b - 1)/(N - b - u + 1).$$

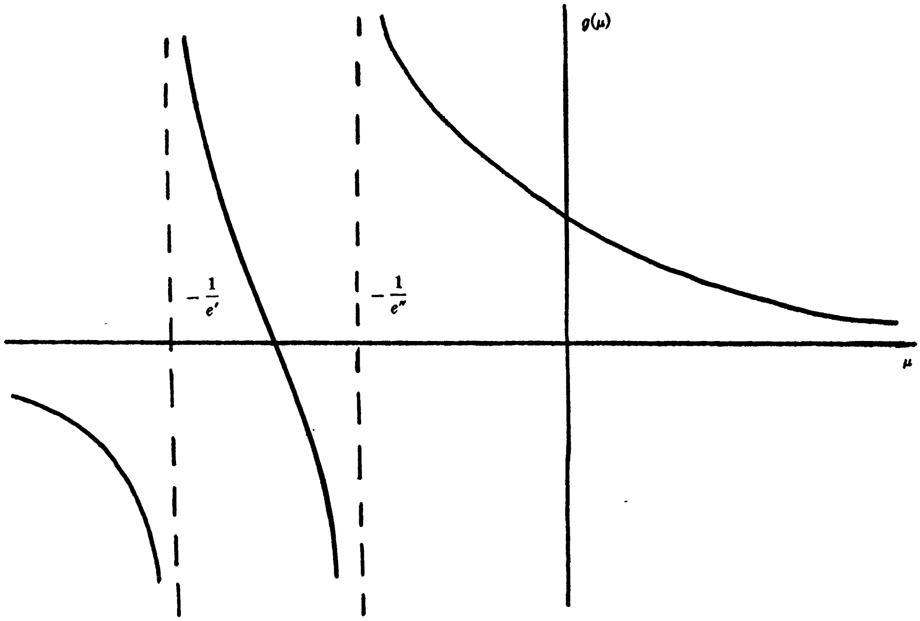
Now

4.4.11 $$\begin{aligned}\textstyle\sum_1[e'' + (e'')^2\mu] + \sum_2[e' + (e')^2\mu] \\ = a_s[e' + (e')^2\mu][e'' + (e'')^2\mu], \qquad s = 1, 2\end{aligned}$$

4.4.12 $$\begin{aligned}a_s e'e''\mu^2 + \left[-\sum_1 \frac{e''}{e'} - \sum_2 \frac{e'}{e''} + a_s(e' + e'')\right]\mu \\ - \frac{\sum_1}{e'} - \frac{\sum_2}{e''} + a_s = 0, \qquad s = 1, 2.\end{aligned}$$

Hence a root of 4.4.9 must also satisy 4.4.12. Now we may see from 4.4.11 that $\mu = -1/e''$ and $\mu = -1/e'$ can not be roots of 4.4.12 and hence we may reverse the steps which brought us from 4.4.9 to 4.4.12 and find that a root of 4.4.12 is a root of 4.4.9.

Denote the left side of 4.4.9 by $g(\mu)$ and graph this function.



Since a_s is non-negative we see that for all $a_s \neq 0$, there are two values of μ that satisfy $g(\mu) = a_s$. Only one of these values may be positive; hence if one of the two roots of equation 4.4.12 is positive, then μ_s is the larger of them. Now write 4.4.12 as follows:

4.4.13 $$b_s\mu^2 + c_s\mu + d_s = 0, \qquad s = 1, 2.$$

Here

$$b_s = a_s e' e'' = a_s \Delta^*$$

4.4.14 $$\begin{aligned} c_s &= a_s(e' + e'') - \textstyle\sum_1 e''/e' - \sum_2 e'/e'' \\ &= a_s(e' + e'') + p'p - m'p(e' + e'') \\ &= (a_s - m'p)H^* + p'p \end{aligned}$$

4.4.15 $$\begin{aligned} d_s &= a_s - \frac{\sum_1}{e'} - \frac{\sum_2}{e''} \\ &= a_s - \frac{(p'p - e''m'p)}{e' - e''} - \frac{(p'p - e'm'p)}{e'' - e'} \\ &= a_s - m'p. \end{aligned}$$

Hence

4.4.16 $$\mu_s = \frac{-c_s + \sqrt{c_s^2 - 4b_s d_s}}{2b_s}, \qquad s = 1, 2;$$

since this is the larger of the two possible roots.

It should be reiterated that the confidence bound in Theorem 1 is given by 4.4.16 only if μ_s is positive, otherwise a zero is substituted for μ_s in the confidence interval.

The test of Theorem 2 is then performed by accepting $\mu \leqq \mu_0$ if

$$4.4.17 \qquad f(\mu_0) = \frac{N - b - u + 1}{b - 1} \frac{1}{\sum Y_j^2} \frac{m'p[1 + \mu_0(e' + e'')] - \mu_0 p'p}{(1 + e'\mu_0)(1 + e''\mu_0)}$$

is less than c and accepting $\mu > \mu_0$ otherwise. Here c is again chosen to be the value of an F variate with $b - 1$ and $N - b - u + 1$ degrees of freedom which has α as its cumulative distribution ordinate.

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