NOTES

A VARIABLE PROBABILITY DISTRIBUTION FUNCTION[1]

By Ruric E. Wheeler

Howard College, Birmingham

1. Introduction and Summary. It is the purpose of this paper to develop an expression for the probability of x successes in n trials, $P(n, x)$, where the probability of success on a single trial depends both on the number of the trial and on the number of previous successes. This result should prove useful in obtaining various probability functions. It will be noted that this work includes the case considered by Woodbury [3].

2. Definitions. Letting $p_{r,s}$ be the probability of a success and $q_{r,s}$ the probability of a failure on the rth trial after s successes, with $p_{r,s} + q_{r,s} = 1$, we formulate the following definition.

Definition 1. We will use the symbol S_i to be a function of $p_{r,s}$ and x, where x is the number of successes, with the following defined properties:

(a) $S_i = \prod_{t=0}^{i-1} p_{t+1,t} \prod_{t=i}^{x-1} p_{t+2,t} q_{i+1,i}$, where $i \leqq x$ ($\prod_i^{i-1}$ is defined to be 1).

(b) The product of S_i, S_j, and S_k in any order (or any number of factors) is defined to be

$$\prod_{t=0}^{i-1} p_{t+1,t} \prod_{t=i}^{j-1} p_{t+2,t} \prod_{t=j}^{k-1} p_{t+3,t} \prod_{t=k}^{x-1} p_{t+4,t} q_{i+1,i} q_{j+2,j} q_{k+3,k},$$

for $i \leqq j \leqq k \leqq x$, where if S_t is a function of x_t successes ($t = i, j, k$), then the quantity x which appears in the formula for the product is the maximum of x_i, x_j and x_k. It should be noted that the product of S_i and S_j is not equal to the value of S_i multiplied by the value of S_j but is given by the above definition.

(c) $S_i^0 = \prod_{t=0}^{i-1} p_{t+1,t}$.

(d) We define $S_i(S_j + S_k)$ to be $S_iS_j + S_iS_k$, where the $(+)$ sign represents ordinary addition.

(e) S_i^r will represent the product $S_iS_iS_i \cdots S_i$ to r factors, and from (b), must be equal $\prod_{t=0}^{i-1} p_{t+1,t} \cdot \prod_{t=i}^{x-1} p_{t+r+1,t} \cdot \prod_{t=1}^{r} q_{t+i,i}$.

(f) Then from (a) and (e), $S_i^m S_j^n$ will be

$$\prod_{t=0}^{i-1} p_{t+1,t} \prod_{t=i}^{j-1} p_{m+t+1,t} \prod_{t=j}^{x-1} p_{m+n+t+1,t} \prod_{t=1}^{m} q_{i+t,i} \prod_{t=1}^{n} q_{m+j+t,j}.$$

Received February 23, 1953; revised October 5, 1955.

[1] These results were included in a dissertation submitted to the University of Kentucky in partial fulfillment of the requirements for the Ph.D. degree, June, 1952.

To illustrate this definition, we consider $S_2^3 S_3^1$, which is

$$\prod_{t=0}^{1} p_{t+1,t} \prod_{t=2}^{2} p_{3+t+1,t} \prod_{t=3}^{x-1} p_{4+t+1,t} \prod_{t=1}^{3} q_{2+t,2} \prod_{t=1}^{1} q_{3+3+t,3}$$

or

$$p_{1,0}\, p_{2,1}\, p_{6,2}\, q_{3,2}\, q_{4,2}\, q_{5,2}\, q_{7,3} \prod_{t=3}^{x-1} p_{t+5,t}\,.$$

We note that the multiplication of these symbols as defined follows the laws of positive integral and zero exponents.

LEMMA 1. *The probability $P(n, x)$ can be expressed as a sum of products of S's for all n and x, such that $n \geqq x$.*

PROOF. Let us consider x successes and $n - x$ failures in the following specified order. Suppose we have α_0 failures, then a success; α_1 failures, then a second success; α_2 failures, then a third success; etc.; finally, the xth success and then α_x failures, where $\alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_x$ must equal $n - x$. By theorems from elementary probability theory, the probability of x successes and $n - x$ failures in this specified order is

$$\prod_{t=1}^{\alpha_0} q_{t,0}\, p_{\alpha_0+1,0} \prod_{t=1}^{\alpha_1} q_{\alpha_0+t+1,1}\, p_{\alpha_0+\alpha_1+2,1} \cdots \prod_{t=1}^{\alpha_x-1} q_{\alpha_0+\alpha_1+\cdots+\alpha_{x-2}+t+x-1,x-1}$$

$$\cdot\, p_{\alpha_0+\cdots+\alpha_{x-1}+x,x-1} \prod_{t=1}^{\alpha_x} q_{\alpha_0+\cdots+\alpha_{x-1}+t+x,x}\,,$$

which we may write in terms of our defined symbols as $S_0^{\alpha_0} S_1^{\alpha_1} S_2^{\alpha_2} \cdots S_x^{\alpha_x}$. Since $P(n, x)$ is the sum of terms such as this, it can be expressed as a sum of products of S's for all n and x such that $n \geqq x$.

3. Development of $P(n, x)$. Let us consider the following partial difference equation

$$P(n, x) = p_{1,0}P''(n - 1, x - 1) + q_{1,0}P'(n - 1, x), \tag{1}$$

where $P''(n - 1, x - 1)$ represents the probability of $x - 1$ successes in $n - 1$ trials with the probability of success on the first of the $n - 1$ trials being $p_{2,1}$, and where $P'(n - 1, x)$ is the probability of x successes in $n - 1$ trials with the probability of success on the first of the $n - 1$ trials being $p_{2,0}$. The boundary conditions for this equation are $P(n, x) = 0$ for $x < 0$, $x > n$, and $P(0, 0) = 1$. Using the generating function $G(x, \theta) = \sum_{k=x}^{\infty} P(k, x)\theta^k$, one may obtain, under the given boundary conditions, a difference equation involving generating functions. From (1) we have that

$$\sum_{k=x}^{\infty} P(k, x)\theta^k = \sum_{k=x}^{\infty} p_{1,0}P''(k - 1, x - 1)\theta^k + \sum_{k=x}^{\infty} q_{1,0}P'(k - 1, x)\theta^k,$$

which in turn gives

$$G(x, \theta) = p_{1,0}\theta G''(x - 1, \theta) + q_{1,0}\theta G(x, \theta). \tag{2}$$

Now let us make use of the displacement operator E. Considering $p_{i,j}$ as a function of i and j, we will use E operating on $p_{i,j}$ to be $p_{i+1,j+1}$. From the properties of E as given in [2],

$$E(S_i) = E\left(\prod_{t=0}^{i-1} p_{t+1,t} \prod_{t=i}^{x-1} p_{t+2,t}\, q_{i+1,i}\right),$$

or

$$E(S_i) = \prod_{t=1}^{i} p_{t+1,t} \prod_{t=i+1}^{x} p_{t+2,t}\, q_{i+2,i+1}\,.$$

Thus, $p_{1,0}E(S_i) = \prod_{t=0}^{i} p_{t+1,t} \prod_{t=i+1}^{x} p_{t+2,t}\, q_{i+2,i+1} = S_{i+1}$, where if S_i is a function of x successes, then S_{i+1} is a function of $x + 1$ successes. Likewise,

$$p_{1,0}E(S_iS_j \cdots S_v) = S_{i+1}S_{j+1} \cdots S_{v+1}$$

and $p_{1,0}E(a_iS_i + a_jS_j + \cdots) = a_iS_{i+1} + a_jS_{j+1} + \cdots$, where the a's are constants.

We will now simplify difference equation (2) by showing that $G''(x, \theta) = EG(x, \theta)$ and $q_{1,0}G'(x, \theta) = S_0G(x, \theta)$. Since these generating functions are power series in θ, it will be sufficient to show that coefficients of like powers of θ in each equation are equal. $P''(n, x)$ has exactly the same form as $P(n, x)$, but has the subscripts of each p and q increased by one (i.e., each $p_{i,j}$ in $P(n, x)$ is $p_{i+1,j+1}$ in $P(n, x)$). From the definition and the properties of E, it is evident that $EP(n, x) = P''(n, x)$. Thus, since $P(n, x)$ and $P''(n, x)$ are the coefficients of θ^n in $G(x, \theta)$ and $G''(x, \theta)$, respectively, we have $EG(x, \theta) = G''(x, \theta)$.

In a like manner, from the properties of S and the definition of $P'(n, x)$, it follows that S_0 multiplied by $P(n, x)$ equals $q_{1,0}P'(n, x)$ and hence $S_0G(x, \theta) = q_{1,0}G'(x, \theta)$.

By making these substitutions, difference equation (2) becomes

$$G(x, \theta) - S_0G(x, \theta)\theta = p_{1,0}EG(x - 1, \theta)\theta$$

or $(1 - S_0\theta)G(x, \theta) = EG(x - 1, \theta)\theta$. Multiplying both sides of this equation by $1 + S_0\theta + S_0^2\theta^2 + \cdots$, we obtain

$$G(x, \theta) = (1 + S_0\theta + S_0^2\theta^2 \cdots)EG(x - 1, \theta)\theta.$$

Then, using $1/(1 - S_0\theta)$ to represent $1 + S_0\theta + S_0^2\theta^2 + \cdots$, we have

$$G(x, \theta) = \frac{\theta}{1 - S_0\theta} EG(x - 1, \theta). \tag{3}$$

Since $G(0, \theta) = 1/(1 - S_0\theta)$, the solution of (3) becomes

$$G(x, \theta) = \frac{\theta^x}{(1 - S_0\theta)(1 - S_1\theta) \cdots (1 - S_x\theta)}. \tag{4}$$

For ordinary multiplication, it is shown on page 313 of [3] that the coefficient of θ^n in an expansion similar to this one is the xth divided difference of S_0^n, a polynomial of degree $n - x$ in the S's. Since our function S satisfies the laws of multiplication for positive integral and zero exponents, the coefficient of θ^n can thus be expressed as the xth divided difference of S_0^n. Or the probability of x successes in n trials is $P(n, x) = \Delta^x S_0^n$, where the symbol Δ is the divided difference.

It is of interest to see what happens to this expression when the probability of success on a single trial depends only on the number of previous successes. Since $P(n, x) = \Delta^x S_0^n$, any term of this polynomial may be written as $S_0^{n_0} S_1^{n_1} \cdots S_x^{n_x}$, where the n_t may take on values $0, 1, 2, \cdots, n - x$. Since our $p_{i,j}$ is now restricted so that it depends only on the number of previous successes (omitting the first subscripts), each term becomes $p_0 p_1 \cdots p_{x-1} q_0^{n_0} q_1^{n_1} \cdots q_x^{n_x}$, or $P(n, x) = p_0 p_1 \cdots p_{x-1} \Delta^x q_0^n$. This is the expression for $P(n, x)$ that was obtained by Woodbury in [3].

By specifying the exact law by which the probability of success on a single trial changes from trial to trial, we may obtain probabilities that determine various desired distributions. As an example, let the probability of success on the first trial be expressed as $p/1$, and let the numerator of this fraction be increased by λ for each success and the denominator increased by λ for each trial. For this distribution, we obtain, from Definition 1,

$$S_0^r = \frac{p}{1 + r\lambda} \frac{p + \lambda}{1 + (r + 1)\lambda} \cdots \frac{p + (x - 1)\lambda}{1 + (x + r - 1)\lambda} \frac{q}{1} \frac{q + \lambda}{1 + \lambda} \cdots \frac{q + (r - 1)\lambda}{1 + (r - 1)\lambda}$$

and

$$S_i^r = \frac{p}{1} \frac{p + \lambda}{1 + \lambda} \cdots \frac{p + (i - 1)\lambda}{1 + (i - 1)\lambda} \frac{p + i\lambda}{1 + (r + i)\lambda} \cdots$$
$$\frac{p + (x - 1)\lambda}{1 + (r + x - 1)\lambda} \frac{q}{1 + i\lambda} \frac{q + \lambda}{1 + (i + 1)\lambda} \cdots \frac{q + (r - 1)\lambda}{1 + (i + r - 1)\lambda}.$$

Since S_0^r is equal to S_i^r for $i = 1, 2, 3, \cdots, x - 1$, and since $\Delta^x S_0^n$ has C_x^n terms, then $P(n, x) = C_x^n S_0^{n-x}$, or

$$C_x^n \frac{p(p + \lambda) \cdots (p + [x - 1]\lambda)(q)(q + \lambda) \cdots (q + [n - x - 1]\lambda)}{(1 + [n - 1]\lambda)!},$$

which is the probability of exactly x successes in n trials for the Polya distribution as given in [1].

REFERENCES

[1] W. FELLER, *Probability Theory and Its Applications*, John Wiley and Sons, New York, 1950, p. 128.

[2] C. JORDAN, *Calculus of Finite Differences*, Chelsea Publishing Co., New York, 1947, pp. 5 and 18.

[3] MAX A. WOODBURY, "On a Probability Distribution", *Ann. Math. Stat.* Vol. 20 (1949), pp. 311–315.