AN EXTENSION OF THE KOLMOGOROV DISTRIBUTION[1, 2]

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1. Summary. Let $x_1, x_2, \cdots, x_n, x'_1, x'_2, \cdots, x'_{nk}$ be independent random variables with a common continuous distribution $F(x)$. Let $x_1, x_2, \cdots, x_n$ have the empiric distribution $F_n(x)$ and $x'_1, x'_2, \cdots, x'_{kn}$ have the empiric distribution $G_{nk}(x)$. The exact values of $P(-y < F_n(s) - G_{nk}(s) < x \text{ for all } s)$ and $P(-y < F(s) - F_n(s) < x \text{ for all } s)$ are obtained, as well as the first two terms of the asymptotic series for large n.

2. Introduction. In a famous paper, Kolmogorov [10] showed that if $F(x)$ is a continuous distribution function, $x_1, x_2, \cdots, x_n, \cdots$ are independent random variables with distribution $F(x)$, and $F_n(x)$ is the empirical distribution based on the variables $x_1, x_2, \cdots, x_n$, then

$$\lim_{n\to\infty} P\left(\sup_{-\infty<x<\infty} |F_n(x) - F(x)| < \frac{\lambda}{n^{1/2}}\right) = \sum_{-\infty}^{\infty} (-1)^j e^{-2j^2\lambda^2}. \tag{1}$$

Since then other proofs [4], [3], [6] have been given, and Chung [2], using the Kolmogorov method of proof, obtained an error term of the order of $n^{-1/10}$ which he then used to obtain a strong limit theorem.

Smirnov obtained a result related to (1) when he showed [13] that

$$\lim_{n\to\infty} P\left(\sup_{-\infty<x<\infty} F_n(x) - F(x) < \frac{\lambda}{n^{1/2}}\right) = 1 - e^{-2\lambda^2}. \tag{1'}$$

Actually Smirnov's results are stronger, since he obtained an exact expression for the probability in (1′) (for finite n) as well as the first two terms of the asymptotic expansion. In an earlier paper, Smirnov showed also that

$$\lim_{n\to\infty} P\left(\sup_{-\infty<x<\infty} |F_{n_1}(s) - G_{n_2}(s)| \leqq \frac{\lambda}{\sqrt{n}}\right) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2\lambda^2} \tag{1''}$$

under the condition $\lambda > 0$, $n = n_1 n_2 / (n_1 + n_2)$ and $n_2 / n_1 = \tau$. (See [13] for further references.)

More recently, Gnedenko, Korolyuk, Rvačeva, and Mihalevič [7], [8], [9], [12] have developed a technique for treating problems of this sort by random-walk methods and have obtained error terms for (1″) under the condition $n_1 = n_2$. We intend in this paper to develop their method further and apply it to obtaining exact expressions for the probabilities appearing in (1) and in (1″) under the condition that τ is an integer. For completeness we are repeating

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some of the work appearing in the above-mentioned papers. Korolyuk has recently published a lengthy paper [11] giving expressions for many of the probabilities we wish to treat. His results, however, differ from ours and indeed are not consistent with earlier-published work (Gnedenko, Doklady 82 (1952), pp. 525–528; also *Math. Nachr.*, Vol. 12 (1954), pp. 29–63).

Our principal results are the following two theorems.

THEOREM 1. *Let $x_1, x_2, \cdots, x_n, x_1', x_2', \cdots, x_{nk}'$ be a sequence of $n(k+1)$ independent random variables with a common continuous distribution. Let $F_n(x)$ and $G_{nk}(x)$ be empiric distributions based on the first n and second kn random variables, respectively. Then*

$$P(-x < G_{nk}(s) - F_n(s) < y \text{ for all } s) = 1 - \binom{(k+1)n}{n}^{-1} \cdot \left\{ \sum_{i=1}^{[kn+\beta/\alpha+\beta]} N(k, n, i(\alpha+\beta) - \beta) + \sum_{i=1}^{[kn+\alpha/\alpha+\beta]} N(k, n, i(\alpha+\beta) - \alpha) - 2 \sum_{i=1}^{[kn/\alpha+\beta]} N(k, n, i(\alpha+\beta)) \right\}$$

if $-1 \leqq -x < y \leqq 1$, where $\alpha = -[-xkn]$, $\beta = -[-ykn]$ and

$$N(k, n, \alpha) = \sum_{j=0}^{[n-\alpha/k]} \frac{\alpha}{(k+1)j+\alpha} \binom{(k+1)j+\alpha}{j} \binom{(k+1)(n-j)-\alpha}{n-j}.$$

THEOREM 2.

$$P(-x < F_n(s) - F(s) < y \text{ for all } s) = 1 - \sum_{i=1}^{[1+y/x+y]} \phi_n(ix + (i-1)y) - \sum_{i=1}^{[1+x/x+y]} \phi((i-1)x + iy) + 2 \sum_{i=1}^{[1/x+y]} \phi_n(ix+iy)$$

for $-1 \leqq x < y \leqq 1$, where

$$\phi_n(x) = \sum_{j=0}^{[n-xn]} \frac{xn}{j+xn} \binom{n}{j} \frac{(j+xn)^j((n-j)-xn)^{n-j}}{n^n}.$$

From these two theorems various limiting relations may be computed.

3. Proofs and corollaries. Suppose given a collection of $n(k+1)$ independent random variables in two sequences:

$$x_1, x_2, \cdots x_n,$$
$$x_1', x_2', \cdots x_{nk}'.$$

Let $F_n(x)$ be the empirical distribution function, continuous on the right, with jumps $1/n$ at $x_1, x_2, \cdots x_n$, and let $G_{nk}(x)$ be the empirical distribution function, continuous on the right, with jumps $1/nk$ at $x_1', x_2', \cdots x_{nk}'$. We introduce the following notation:

$$D^+ = \sup_x(G_{nk}(x) - F_n(x)),$$
$$D^- = -\inf_x(G_{nk}(x) - F_n(x)) = \sup_x(F_n(x) - G_{nk}(x)).$$

The method used will involve finding the joint distribution of D^+ and D^- and then taking the limit as $k \to \infty$. This will provide a proof of Theorem 1 and Theorem 2.

Gnedenko and Korolyuk [7] introduced a technique for finding the joint distribution of D^+ and D^- by considering a related random-walk problem. Order the x_i and x_i' random variables in order of their numerical value and call the new sequence

$$z_1, z_2, \cdots z_{n(k+1)}.$$

Let

$$\sigma_j = \begin{cases} +1 & \text{if } z_j \text{ is from } x_1', x_2', \cdots x_{nk}', \\ -k & \text{if } z_j \text{ is from } x_1, x_2, \cdots x_n, \end{cases}$$

and

$$S_p = \sum_1^p \sigma_i. \qquad \text{for} \quad p = 1, 2, \cdots.$$

Then it is not difficult to see that

$$(2) \quad P(D^- < y, D^+ < x) = P(-\beta < S_j < \alpha, \qquad j = 1, 2, \cdots (k+1)n),$$

where $\alpha = -[-xkn]$ and $\beta = -[-ykn]$. This reduces the problem to one of investigating a linear random walk with $(k+1)n$ steps which starts at 0, moves at each step either one unit to the right or k units to the left, and ends after $(k+1)n$ steps at 0 again. In [7], [8] the investigation was carried out for $k = 1$, although the authors were apparently unaware that the $k = 1$ case had been treated extensively by Bachelier [1] in connection with certain gambler's-ruin problems. Some results in the $k > 1$ case were obtained in [9]. Because of the independence of $x_1, x_2, \cdots, x_n, x_1', x_2', \cdots, x_{kn}'$, each path is equally likely, so that we essentially have only to count paths.

Divide the class of all paths $\mathfrak{M}$ into the following nonintersecting sets:

$\mathfrak{A}_0$: paths reaching neither $-\beta$ nor α.

$\mathfrak{A}_1$: paths reaching α but not $-\beta$.

$\mathfrak{A}_2$: paths first reaching α (i.e., before reaching $-\beta$), then reaching at some subsequent step $-\beta$, but not thereafter reaching α.

$\mathfrak{A}_3$: paths first reaching α, then $-\beta$, then α, but not thereafter reaching $-\beta$.

etc.

The classes $\mathfrak{B}_i$, $i = 1, 2, \cdots$ are defined in the same way with $-\beta$ and α interchanged. For k and n fixed, the classes $\mathfrak{A}_i$ and $\mathfrak{B}_i$ will be empty for i sufficiently large. Also,

$$\mathfrak{M} = \mathfrak{A}_0 + \textstyle\sum_i(\mathfrak{A}_i + \mathfrak{B}_i).$$

The classes A_i are defined as follows:

A_1 : paths reaching α at least once, regardless of what happens at any other step.

A_2 : paths reaching α and $-\beta$ at least once in the order $\alpha, -\beta$, regardless of what happens at any other step.

A_3 : paths reaching α and $-\beta$ at least once in the order $\alpha, -\beta, \alpha$, regardless of what happens at any other step.

etc.

The classes B_i are defined in the same way with α and $-\beta$ interchanged. Because of the equalities

$$A_1 = \mathcal{A}_1 + \sum_{i=2}^{\infty} (\mathcal{A}_i + \mathcal{B}_i), \qquad B_1 = \mathcal{B}_1 + \sum_{i=2}^{\infty} (\mathcal{A}_i + \mathcal{B}_i),$$

$$A_2 = \mathcal{A}_2 + \sum_{i=3}^{\infty} (\mathcal{A}_i + \mathcal{B}_i), \qquad B_2 = \mathcal{B}_2 + \sum_{i=3}^{\infty} (\mathcal{A}_i + \mathcal{B}_i),$$

etc., we have for arbitrary $i \geqq 1$

$$A_{2i-1} + B_{2i-1} - A_{2i} - B_{2i} = \mathcal{A}_{2i-1} + \mathcal{B}_{2i-1} + A_{2i} + \mathcal{B}_{2i},$$

so that

$$\mathcal{A}_0 = \mathcal{M} - \sum_{i=1}^{\infty} (A_{2i-1} + B_{2i-1} - A_{2i} - B_{2i}).$$

Since $A_{2i-1} - A_{2i}$ and $B_{2i-1} - B_{2i}$ are disjoint, $A_{2i-1} \supset A_{2i}$, and $B_{2i-1} \supset B_{2i}$, we have

$$N(\mathcal{A}_0) = N(\mathcal{M}) - \sum_{i=1}^{\infty} (N(A_{2i-1}) + N(B_{2i-1}) - N(A_{2i}) - N(B_{2i})),$$

where $N(A)$ is the cardinality of A. This formula was obtained in both [1] and [8], although the computation of the number of paths in the classes A_i, B_i was carried out only for $k = 1$, in which case a reflection principle will work. Let $N(k, n, \alpha)$ be the number of paths in the class A_1. We now show that the number of paths in A_2 is $N(k, n, \alpha + \beta)$ by mapping the class A_2 in a 1:1 manner on the class of paths which cross $\alpha + \beta$ at least once. Note that if a path crosses α, it actually reaches the point α, since all steps to the right have length one. Also, if $-\beta$ is crossed from the right, the path must reach $-\beta$ on the subsequent crossing from the left for the same reason. Divide the steps in an A_2 path into four parts ρ_1, ρ_2, ρ_3, ρ_4. ρ_1 consists of those steps from the first to the first step reaching α. ρ_2 consists of those steps from the first after ρ_1 to the first step actually ending at $-\beta$. ρ_3 consists of those from the first after ρ_2 to the first step ending at 0, and ρ_4 consists of the remainder. The path with steps in the order ρ_1, ρ_3, ρ_2, ρ_4 then crosses $\alpha + \beta$. Moreover, this path reaches α for the first time at the end of the ρ_1 steps, reaches $\alpha + \beta$ for the first time at the end of the ρ_3 steps, and reaches 0 again for the first time at the end of the ρ_2 steps. From this we conclude that the original A_2 path can be reconstructed from its image. Since the inverse mapping takes every path crossing $\alpha + \beta$ into a path from A_2, we find that the mapping is 1:1.

Using the same idea, the class A_3 can be put in 1:1 correspondence with the class of paths crossing $2\alpha + \beta$ at least once. In general,

$$n\{A_{2i-1}\}\begin{cases} = N(k, n, i(\alpha+\beta)-\beta) & \text{if } i(\alpha+\beta)-\beta \leqq kn, \\ = 0 & \text{if } i(\alpha+\beta)-\beta > kn. \end{cases}$$

$$n\{B_{2i-1}\}\begin{cases} = N(k, n, i(\alpha+\beta)-\alpha) & \text{if } i(\alpha+\beta)-\alpha \leqq kn, \\ = 0 & \text{if } i(\alpha+\beta)-\alpha > kn. \end{cases}$$

$$n\{A_{2i}\}\begin{cases} = n\{B_{2i}\} = N(k, n, i(\alpha+\beta)) & \text{if } i(\alpha+\beta) \leqq kn, \\ = 0 & \text{if } i(\alpha+\beta) > kn. \end{cases}$$

Since the total number of paths is $\binom{(k+1)n}{n}$, we therefore find

$$P(D^- < y, D^+ < x)$$

$$(3)\quad = 1 - \binom{(k+1)n}{n}^{-1}\left\{\sum_{i=1}^{[(kn+\beta)/(\alpha+\beta)]} N(k,n,i(\alpha+\beta)-\beta) + \sum_{i=1}^{[(kn+\alpha)/(\alpha+\beta)]} \cdot N(k,n,i(\alpha+\beta)-\alpha) - 2\sum_{i=1}^{[(kn)/[\alpha+\beta)]} N(k,n,i(\alpha+\beta))\right\}.$$

if $-1 \leqq -x < y \leqq 1$ where $\alpha = -[-xkn]$, $\beta = -[-ykn]$.

The computation for $N(k, n, \alpha)$ which follows is based on the work of Bachelier ([1], pp. 101–103). The author is indebted to Dr. Warren Hirsch for pointing out that the basic argument given for $k = 1$ was to be found there and could be extended to $k > 1$.

Observe that the paths cannot cross the point α before α steps, and that in general a crossing can occur only after $\alpha + (k + 1)i$ steps where $i = 0, 1, 2, \cdots [n - \alpha/k]$. The upper limit on i is required by the fact that after the last possible crossing the path must still have enough steps left to return to the origin. Let M_i be the total number of paths which cross at the $\alpha + (k + 1)i$ step and let $\bar{M}_i$ be the number of paths which cross at $\alpha + (k + 1)i$, but which have crossed at some earlier step. In order to cross at the $\alpha + (k + 1)i$ step, there must be $\alpha + ki$ positive steps to that point and i negative ones. Let T_i denote the total number of ways of combining these $\alpha + ki$ positive and i negative steps in such a manner that the crossing at $\alpha + (k + 1)i$ is the first crossing. Let M_i' denote the number of paths which at the $\alpha + (k + 1)i - 1$ step are at $\alpha + k$, and which at the $\alpha + (k + 1)i$ step cross the point α.

In order for a path to be counted in either $\bar{M}_j$ or M_j', it must cross α for the first time at one of the steps $\alpha + (k + 1)i$ for $i < j$. Using this it is easy to see that

$$\bar{M}_j = \sum_{i<j} T_i \binom{(k+1)(j-1)}{j-1}\binom{(k+1)(n-j)-\alpha}{n-j}$$

and

$$M'_j = \sum_{i<j} T_i \binom{(k+1)(j-1)-1}{j-i-1}\binom{(k+1)(n-j)-\alpha}{n-j}.$$

Since

$$\binom{(k+1)(j-i)}{j-i}\binom{(k+1)(j-i)-1}{j-i-1}^{-1} = k+1, \qquad \text{we have}$$

$$\bar{M}_j = (k+1)M'_j.$$

However, M'_j can be computed directly, since of the first $(k+1)j+\alpha-1$ steps $j-1$ must be negative, and of the remaining $(k+1)(n-j)-\alpha$ steps (after the negative step which takes the path to the point α) there must be $n-j$ negative:

$$M'_j = \binom{(k+1)j+\alpha-1}{j-1}\binom{(k+1)(n-j)-\alpha}{n-j}.$$

Therefore, since

$$M_j = \binom{(k+1)j+\alpha}{j}\binom{(k+1)(n-j)-\alpha}{n-j},$$

$$M_j - \bar{M}_j = \left[\binom{(k+1)j+\alpha}{j} - (k+1)\binom{(k+1)j+\alpha-1}{j-1}\right] \cdot\binom{(k+1)(n-j)-\alpha}{n-j}$$

$$= \frac{\alpha}{(k+1)j+\alpha}\binom{(k+1)j+\alpha}{j}\binom{(k+1)(n-j)-\alpha}{n-j}.$$

Since $M_j - \bar{M}_j$ is the number of paths which cross at the $(k+1)j+\alpha$ step for the first time,

$$N(k,n,\alpha) = \sum_{j=0}^{[n-\alpha/k]} \frac{\alpha}{(k+1)j+\alpha}\binom{(k+1)j+\alpha}{j}\binom{(k+1)(n-j)-\alpha}{n-j}.$$

This proves Theorem 1.

We now investigate $\lim_{k\to\infty}\binom{(k+1)n}{n}^{-1}N(k,n,i\alpha+p\beta)$. A straightforward calculation shows that

$$\lim_{k\to\infty}\binom{(k+1)n}{n}^{-1}\binom{(k+1)j+i\alpha+p\beta}{j}\binom{(k+1)(n-j)-i\alpha-p\beta}{n-j}$$

$$= \binom{n}{j}\frac{(j+ixn+pyn)^j((n-j)-ixn-pyn)^{n-j}}{n^n}.$$

Therefore,

$$\lim_{k\to\infty}\binom{(k+1)n}{n}^{-1} N(k,n,i\alpha+p\beta)$$

$$(4) \qquad = \sum_{j=0}^{[n-ixn-pyn]} \frac{ixn+pyn}{j+ixn+pyn}\binom{n}{j} \cdot \frac{(j+ixn+pyn)^j((n-j)-ixn-pyn)^{n-j}}{n^n}$$

$$= \phi_n(ix+py).$$

From (2), (3), and (4), using the fact that $\lim_{k\to\infty} G_{nk}(x) = F(x)$ uniformly with probability one (Glivenko-Cantelli Theorem [5], p. 260), we see that

$$\lim_{k\to\infty} P(D^- < y, D^+ < x) = \lim_{k\to\infty} P(-y < G_{nk}(s) - F_n(s) < x \text{ for all } s)$$

$$= P(-y < F(s) - F_n(s) < x \text{ for all } s)$$

$$= 1 - \sum_{i=1}^{[(1+y)/(x+y)]} \phi_n(ix+(i-1)y) - \sum_{i=1}^{[(1+x)/(x+y)]} \cdot\phi_n((i-1)x+iy) + 2\sum_{i=i}^{[1/(x+y)]} \phi_n(ix+iy).$$

This completes the proof of Theorem 2.

It should be noted that

$$P(F(s) - F_n(s) < x \text{ for all } s) = 1 - \phi_n(x),$$

and that this is exactly the result obtained by Smirnov [13]. In this paper Smirnov also outlines the technique whereby it may be shown that under the conditions $x > x_0 > 0$ and $x^3/\sqrt{n} = o(1)$,

$$\phi_n\left(\frac{x}{\sqrt{n}}\right) = e^{-2x^2}\left(1 - \tfrac{2}{3}\frac{x}{\sqrt{n}} + O\left(\frac{x^2}{n}\right)\right).$$

Combining this result with Theorem 2, we obtain the following corollary.

COROLLARY 1. *If* $x > x_0 > 0$, $y > y_0 >, 0\ x^3/\sqrt{n} = o(1)$, *and* $y^3/\sqrt{n} = o(1)$, *then*

$$P(-y/n^{1/2} < F(s) - F_n(s) < x/n^{1/2} \text{ for all } s)$$

$$= 1 - \sum_{i=1}^{\infty} \{e^{-2(ix+(i-1)y)^2} + e^{-2((i-1)x+iy)^2} - 2e^{-2(ix+iy)^2}\}$$

$$+ \frac{1}{n^{1/2}}\tfrac{2}{3}\sum_{i=1}^{\infty}\{(ix+(i-1)y)e^{-2(ix+(i-1)y)^2}$$

$$+ ((i-1)x+iy)^2 e^{-2(i-1)x+iy)^2}$$

$$- 2(ix+iy)e^{-2(ix+iy)^2}\} - O\left[\frac{1}{n}\sum_{1}^{\infty}\{(ix+(i-1)y)^2 e^{-2(ix+(i-1)y)^2}\right.$$

$$\left. + ((i-1)x+iy)^2 e^{-2((i-1)x+iy)^2} - 2(ix+iy)^2 e^{-2(ix+iy)^2}\}\right].$$

Using similar techniques one may take the limit as $n \to \infty$ in the result of Theorem 1. The computations become more complicated, and we will state only a somewhat weaker version than can actually be obtained.

COROLLARY 2. *For fixed x and y and k an integer,*

$$P\left(-\frac{y\sqrt{k+1}}{\sqrt{kn}} < G_{nk}(s) - F_n(s) < \frac{x\sqrt{k+1}}{\sqrt{kn}} \text{ for all } s\right)$$

$$= 1 - \sum_{i=1}^{\infty} \{e^{-2(ix+(i-1)y)^2} + e^{-2((i-1)x+iy)^2} - 2e^{-2(ix+iy)^2}\} + \frac{1}{\sqrt{k(k+1)n}}$$

$$\cdot \sum_{i=1}^{\infty} \left\{(ix + (i-1)y)\left(\tfrac{2}{3}(1-k) - \frac{4Q(\sqrt{n}\sqrt{k(k+1)}(ix+(i-1)y))}{\sqrt{k(k+1)}}\right)\right.$$

$$\cdot e^{-2(ix+(i-1)y)^2} + ((i-1)x + iy)$$

$$\cdot \left(\tfrac{2}{3}(1-k) - \frac{4Q(\sqrt{n}\sqrt{k(k+1)}((i-1)x+iy)^2)}{\sqrt{k(k+1)}}\right) e^{-2((i-1x)+iy)^2}$$

$$\left. - 2(ix+iy)\left(\tfrac{2}{3}(1-k) - \frac{4Q(\sqrt{n}\sqrt{k(k+1)}(ix+iy)^2)}{\sqrt{k(k+1)}}\right) e^{-2(ix+iy)^2}\right\} + O\left(\frac{1}{n}\right),$$

where the function $Q(x)$ is defined by the equation

$$-[-x] = x + Q(x).$$

The special case $k = 1$ is treated in [7].

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